An Analytical Method for Bit Error Probability Determination in Multi-Rate DS-WCDMA Systems

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Abstract—One of the main objectives of third generation cellular systems is to support several types of applications with different requirements on bit rate and performance, such as speech, data and video services. On investigating multi-rate systems it is important to be able to evaluate their performance so that their capability of supporting some application requirements can be verified. In this paper, an analytical method for bit error probability determination in multi-rate DS-WCDMA cellular systems is proposed based on Gaussian approximations for multiple access interference. A mathematical model which represents interference from neighbour cells as a function of path loss model, mobile user densities on cells and cellular architecture is also presented.

Keywords — bit error probability, multi-rate, CDMA, AWGN, multiple access interference.

I. INTRODUCTION

The convergence between data services and mobility is a key step in the evolution of communication systems, as proposed by third generation cellular systems. One of the main objectives of these emerging systems is to support simultaneously several types of applications with different requirements on bit rate and performance, such as speech, data and video services. On investigating multi-rate systems [1] [2], it is important to be able to evaluate their performance so that their capability of supporting some application requirements can be verified. This paper addresses bit error probability of multi-rate Direct Sequence Wideband Code Division Multiple Access (DS-WCDMA) cellular systems in Additive White Gaussian Noise (AWGN) channels, based on [3]’s approach.

This paper is organized as follows. Section II provides a description of the reverse link of a multi-rate DS-WCDMA cellular system where each cell supports several applications with different requirements on bit rate and bit error probability. In section III multiple access interference (MAI) is characterized by its mean and variance, assuming Gaussian approximation. A mathematical model which represents MAI from neighbour cells as a function of path loss model, mobile user densities on cells, and cellular architecture and geometry is also presented. In section IV, a bit error probability expression for AWGN channel assuming QPSK modulation is derived. Finally, numerical results are presented in section V and conclusions are drawn in section VI.

II. MULTI-RATE CELLULAR SYSTEM MODEL

The reverse link of a multi-rate DS-WCDMA cellular system is represented in Fig.1. For convenience, the system is modelled as a group of subsystems, each one representing a specific application. In this model, when an user develops an application, it is characterized as an user of a specific subsystem, what means that some transmitter and receiver parameters are adjusted in a way that it can support required bit rate and performance. According to Fig. 1, subsystem i from cell g has $U_{gi}$ users and supports bit rate $R_{gi}$ and bit error probability $P_{E_{gi}}$ specified for its correspondent application.

The multi-rate transmitter $g_{ik}$ translates each group of $log_{2}M_{gi}$ bits of the binary sequence from user k in subsystem i of a cell g into a complex valued symbol according to a mapping rule. It generates a symbol sequence $A_{gi}(m) = A^i_{gi}(m) + jA^0_{gi}(m)$, which modulates the amplitude of rectangular pulses with unitary amplitude and duration $T_{gi} = \log_{2}M_{gi}/R_{gi}$. The resulting information signals $b^i_{gi}(t)$ and $b^0_{gi}(t)$ are multiplied by spreading codes $c^i_{gi}(t)$ and $c^0_{gi}(t)$, which are periodic signals consisting of rectangular pulses (chips) with a constant duration $T_{c}$ assumed the same in all subsystems. The resulting signals modulate the amplitude and phase of a carrier with central frequency $\omega_{c}$, amplitude $\sqrt{2P_{gi}}$ and phase $\theta_{gi}$, where $P_{gi}$ is adjusted according to power control requirements and depends on the frequency band available for the mobile cellular service. Thus, the transmitted signal for user $g_{ik}$ is

$$s_{gi}(t) = \sqrt{2P_{gi}} b^i_{gi}(t)c^i_{gi}(t) \cos(\omega_{c}t + \theta_{gi}) - \sqrt{2P_{gi}} b^0_{gi}(t)c^0_{gi}(t) \sin(\omega_{c}t + \theta_{gi}).$$ (1)

The signal provided by the AWGN channel to receivers located at the base station in cell $h$, assuming asynchronous transmission, is

$$r_{ih}(t) = n(t) + \sum_{g=1}^{G} \sum_{i=1}^{U_{gi}} \sum_{k=1}^{M_{gi}} s_{gi}(t - \tau_{gik})$$ (2)
where \( n(t) \) is a zero-mean additive white Gaussian random process with two-sided spectral density \( N_0/2 \); \( Y \) is the number of cells in the multi-rate system; \( X_\rho \) is the number of subsystems in a cell \( g \); \( U_{g,i} \) is the number of users in subsystem \( g; \) \( \alpha_{gik} \) is the attenuation factor and \( \tau_{gik} \) is the propagation delay that the channel introduces into signal \( s_{gik}(t) \) at the path between the transmitter \( g \) and base station in cell \( h \), modelled respectively as a deterministic time-invariant variable and a time-invariant random variable uniformly distributed over zero and \( T_r \). The multi-rate receiver \( h jl \) detects the binary sequence sent by user \( h jl \) based on a decision variable obtained from despread and demodulation of signal \( r_{h jl}(t) \). The components of the decision variable \( \hat{A}_{h jl} = \hat{A}_{h jl}(m_{h jl}) \) at the multi-rate receiver \( h jl \), assuming that \( \tau_{gik}^{(m)} = 0 \) and \( \phi_{gik}^{(m)} = 0 \) are references for the other propagation delays \( \tau_{gik}^{(h)} \) and phases \( \phi_{gik}^{(h)} \) are got from (1) and (2) as

\[
\hat{A}_{h jl} = \int r_{h jl}(t) c_{h jl}^{(m)}(t) \cos \omega t \, dt = \eta_{h jl} + \beta_{h jl} + \gamma_{h jl}^{(m)} \quad (3)
\]

and

\[
\hat{A}_{h jl} = - \int r_{h jl}(t) c_{h jl}^{(m)}(t) \sin \omega t \, dt = \eta_{h jl} + \beta_{h jl} + \gamma_{h jl}^{(m)} \quad (4)
\]

where \( \eta_{h jl} \) and \( \eta_{h jl}^{(m)} \) are the zero-mean Gaussian noise terms with variance \( N_0/4 \); \( \beta_{h jl} \) and \( \beta_{h jl}^{(m)} \) are the terms which contain the information sent by user \( h jl \); and \( \gamma_{h jl} \) and \( \gamma_{h jl}^{(m)} \) represent the multiple access interference. These variables are given as follows:

\[
\eta_{h jl}^{(m)} = \int n(t) c_{h jl}^{(m)}(t) \cos \omega t \, dt ;
\]

\[
\eta_{h jl} = \int n(t) c_{h jl}^{(m)}(t) \sin \omega t \, dt ;
\]

\[
\beta_{h jl}^{(m)} = \sqrt{P_{h jl}/2} \alpha_{h jl}^{(m)} A_{h jl}(m_{h jl}) T_{h jl} ;
\]

\[
\beta_{h jl} = \sqrt{P_{h jl}/2} \alpha_{h jl} A_{h jl}(m_{h jl}) T_{h jl} ;
\]

\[
\gamma_{h jl} = \sum_{g=1}^{G} \sum_{k=1}^{K} \sum_{i=1}^{I} \sqrt{P_{h jl}/2} \alpha_{gik}^{(h)} \left[ \psi_{gik,hj}(t) \cos \phi_{gik,hj}(t) - \psi_{gik,hj}(t) \sin \phi_{gik,hj}(t) \right];
\]

\[
\gamma_{h jl}^{(m)} = \sum_{g=1}^{G} \sum_{k=1}^{K} \sum_{i=1}^{I} \sqrt{P_{h jl}/2} \alpha_{gik}^{(m)} \left[ \psi_{gik,hj}(t) \sin \phi_{gik,hj}(t) + \psi_{gik,hj}(t) \cos \phi_{gik,hj}(t) \right];
\]

with

\[
\psi_{gik,hj}(t) = \int b_{gik}(t - \tau_{gik}^{(h)}) c_{h jl}^{(m)}(t - \tau_{gik}^{(m)}) \, dt
\]

where \( H_1 \) and \( H_2 \) can be I or Q, denoting In-phase or Quadrature-phase terms, respectively.

### III. MULTIPLE ACCESS INTERFERENCE

Assuming that the multiple access interference components \( \gamma_{h jl} \) and \( \gamma_{h jl}^{(m)} \) are Gaussian random variables, their behaviour are completely described by their mean and variance. Although
this assumption is not necessarily correct because the number of interfering users is finite, Gaussian approximation has been shown to be relatively good if the number of interfering users is large so Central Limit Theorem can be invoked [5]. Therefore, we determine mean and variance of \( v_{ij}^t \) in this section. From analogous assumptions and derivations it can be shown that the results for \( v_{ij}^t \) are the same.

\[ A. \text{ Mean} \]

It is easy to see from (9) that if random variables \( \phi_{ik}^t \), \( \psi_{ik,jh}^t \) and \( \psi_{jk,ih}^t \) are independent and phases \( \theta_{ijk}^t \) are uniformly distributed over 0 and 2\( \pi \), then \( \mathbb{E}[v_{ij}^t] = 0 \) because \( \mathbb{E}[\cos\theta_{ijk}^t] \) and \( \mathbb{E}[\sin\theta_{ijk}^t] \) are null.

\[ B. \text{ Variance} \]

The variance of \( v_{ij}^t \) is determined by the probability distributions of \( b_{ik}^+(t) \), \( b_{ik}^-(t) \), \( c_{ik}^+(t) \), \( c_{ik}^-(t) \), and \( \tau_{ik}^t \), based on the assumption that these random variables in signals from one or various users are independent.

Since phases \( \theta_{ijk}^t \) are uniformly distributed over 0 and 2\( \pi \), we have from (9) that

\[
\text{Var}\left[v_{ij}^t\right] = 4 \sum_{j=1}^{N} \sum_{k=1}^{N} P_{ik} \left( \alpha_{ijk}^t \right)^2 \mathbb{E}[\psi_{ik,jh}^t]^2 + \mathbb{E}[\psi_{jk,ih}^t]^2] / 4. \tag{12}
\]

Then, let us evaluate \( \mathbb{E}[\psi_{ik,jh}^t] \). For convenience \( \psi_{ik,jh}^t \) can be written as the sum of integrals over the intervals where \( b_{ik}^+(t) \) is constant. According to Fig. 2 and (11) we get

\[
\psi_{ik,jh}^t = b_{ik}^+(t_2 - T_D) R_{ik,jh}^t(t_2,t_2)
+ \sum_{p=1}^{N} b_{ik}^+(t_2 + pT_D) R_{ik,jh}^t(t_2 + pT_D,t_2 + (p + 1)T_D) \tag{13}
+ b_{ik}^-(t_{w-1}) R_{ik,jh}^t(t_{w-1},t_{w-1})
\]

where \( H = (t_{w-1} - t_2)|T_D| - 1 \) and

\[
R_{ik,jh}^t(t_2,t_2) = \int_{t_2}^{t_2} c_{ik}^+(t - \tau_{ik}^t) k_{ih}^t(t) dt. \tag{14}
\]

Observe that (13) describes the three possible cases of relation between bit rates of users \( ik \) and \( jh \): if \( T_D < \tau_{ik}^t \) then \( t_2 = mT_D + \tau_{ik}^t \) and \( t_{w-1} = t_2 + \left( T_D - \tau_{ik}^t \right)/T_D \); if \( T_D = \tau_{ik}^t \) then \( t_2 = t_{w-1} = mT_D + \tau_{ik}^t \) and the second term in (13) is null; and if \( \tau_{ik}^t > T_D \) then \( t_2 = t_{w-1} = mT_D + \tau_{ik}^t \) and the second term in (13) is null, or \( t_w = t_{w-1} = t_2 = (m_0 + 1)T_D \) and second and third terms in (13) are null.

Assuming that \( b_{ik}^+(t) \) is a stationary random process, its symbols are equiprobable and zero-mean, and consecutive symbols are independent, we have from (13) that

\[
\mathbb{E}[\psi_{ik,jh}^t] = \mathbb{E}\left[\left( b_{ik}^+(t_2) \right)^2 \right] \mathbb{E}\left[ \sum_{p=0}^{N-1} \left( R_{ik,jh}^t(t_2 + pT_D) + R_{ik,jh}^t(t_2 + pT_D,t_2 + (p + 1)T_D) \right) \right]. \tag{15}
\]

Then, assuming \( c_{ik}^+(t) \) and \( c_{ik}^-(t) \) are stationary random processes with equiprobable zero-mean chips, consecutive chips are independent, and also observing that \( R_{ik,jh}^t(t_2 + pT_D) \) can be written as the sum of integrals over the intervals where both \( c_{ik}^+(t) \) and \( c_{ik}^-(t - \tau_{ik}^t) \) are constant, we have

\[
\mathbb{E}\left[ \sum_{p=0}^{N-1} \left( R_{ik,jh}^t(t_2 + pT_D) + R_{ik,jh}^t(t_2 + pT_D,t_2 + (p + 1)T_D) \right) \right] = \sum_{p=0}^{N-1} \mathbb{E}\left[ \left( c_{ik}^+(t) \right)^2 \right] \mathbb{E}\left[ T - \tau_{ik}^t + K_{ik}^t(T_D) \right]. \tag{16}
\]

where \( K_{ik}^t \) is a positive integer constant that satisfies \( 0 \leq \tau_{ik}^t - K_{ik}^t(T_D) < T_D \) and \( N = T_D/T_D \) is the processing gain of subsystem \( hj \), as illustrated in Fig. 2.

Finally, assuming propagation delays \( \tau_{ik}^t \) are random variables uniformly distributed over 0 and \( T_D \) and observing that \( \mathbb{E}\left[ \left( \tau_{ik}^t - K_{ik}^t(T_D) \right)^2 + \left( T_D - \tau_{ik}^t + K_{ik}^t(T_D) \right)^2 \right] \) corresponds to the sum of expectations on intervals from \( uT_D \) to \( (u+1)T_D \) where \( K_{ik}^t = uT_D \) we obtain

\[
\mathbb{E}\left[ \left( \tau_{ik}^t - K_{ik}^t(T_D) \right)^2 + \left( T_D - \tau_{ik}^t + K_{ik}^t(T_D) \right)^2 \right] = 2T_D^2/3. \tag{17}
\]

From (15), (16) and (17) we determine \( \mathbb{E}\left[\psi_{ik,jh}^t\right] = 2N^2T_D^2\mathbb{E}\left[\left( b_{ik}^+(t) \right)^2\right]/3 \) and from analogous assumptions and derivations it can be shown that \( \mathbb{E}\left[\psi_{jk,ih}^t\right] = 2N^2T_D^2\mathbb{E}\left[\left( b_{ik}^+(t) \right)^2\right]/3 \). Thus, from (12) we get

\[ \mathbb{E}[\psi_{ik,jh}^t]^2 + \mathbb{E}[\psi_{jk,ih}^t]^2] = 2N^2T_D^2\mathbb{E}[\left( b_{ik}^+(t) \right)^2]/3. \]

Figure 2. Chip and symbol durations
Cellular Environment. The power on the base station in cell between transmitter and receiver, where $C$. Interference Power

$$S_{dMl}^{(h)} = \frac{\sum_{k} S_{p}^{(e)}}{\sum_{k} S_{p}^{(e)}} \frac{\gamma}{6} \quad (18)$$

where $S_{dMl}^{(h)}$ is the total average interference power from all interfering users on receiver $hjl$, which is discussed as follows.

C. Interference Power

Consider a path loss model where the relation between average transmitted power and average received power is inversely proportional to the $\gamma$-th power of the distance between transmitter and receiver, where $\gamma$ depends on the cellular environment. The power on the base station in cell $h$ received from all $U_{gi}$ users of subsystem $i$ in a cell $g$ is given by

$$S_{p}^{(b)} = \sum_{k=1}^{U_{gi}} \frac{S_{p}^{(e)}}{k} \left( \frac{y_{(\sigma_{g})}^{(s)}}{\sqrt{(y_{(\sigma_{g})}^{(s)})^2 + (d_{g,h})^2 + 2 d_{g,h} y_{(\sigma_{g})}^{(s)} \cos \phi_{(\sigma_{g})}^{(s)}}} \right)^{\gamma} \quad (19)$$

where $y_{(\sigma_{g})}^{(s)}$, $\phi_{(\sigma_{g})}^{(s)}$ and $d_{g,h}$ are defined in Fig. 3, and $S_{dMl}^{(h)}$ is the power on cell $g$ received from an user $gik$, which must be the same for all $U_{gi}$ users in subsystem $g$ since they have to support the same application requirements.

As positions of users in cell $g$ is seldom known due to their mobility, it is convenient to define $S_{dMl}^{(h)}$ as the sum of powers from $dU_{gi}$ users of subsystem $gi$, which are in a region with area $dA$ in cell $g$, that is to say, $S_{dMl}^{(h)} = S_{p}^{(b)}$, where $S_{p}^{(b)}$ is the power on cell $g$ received from all users of subsystem $gi$;

$$w_{p}^{(b)} = \frac{1}{U_{gi}} \int_{area} \left( \frac{y_{(\sigma_{g})}^{(s)}}{\sqrt{(y_{(\sigma_{g})}^{(s)})^2 + (d_{g,h})^2 + 2 d_{g,h} y_{(\sigma_{g})}^{(s)} \cos \phi_{(\sigma_{g})}^{(s)}}} \right)^{\gamma} \rho_{g} \ dA \quad (20)$$

is a function that characterizes how the total power from users of subsystem $gi$ interferes to a neighbour cell $h$, and $\rho_{g} = dU_{gi}/dA$ is the user density in subsystem $gi$. If a cell $g$ is circular with radius $\lambda_{c}$, its base station is located at its centre, and the $U_{gi}$ users in subsystem $gi$ are uniformly distributed over its area, then $\rho_{g} = U_{gi}/\pi \lambda_{c}^2$. Assuming $\gamma = 4$, we get from (20) that [6]

$$w_{p}^{(h)} = 4 d_{g,h}^2 \lambda_{c} \ln \left( \frac{d_{g,h}^2}{\lambda_{c}^2} \right) - 6 d_{g,h}^2 - \lambda_{c}^2 + \lambda_{c}^2 \left( \frac{d_{g,h}^2}{\lambda_{c}^2} - \lambda_{c}^2 \right) \quad (21)$$

Defining

$$w_{g}^{(b)} = \left( \sum_{k} S_{p}^{(e)} w_{g}^{(b)} \right) / \sum_{k} S_{p}^{(e)} \quad (22)$$

as a function that characterizes how the total power from all users of cell $g$ interferes to a neighbour cell $h$, we see from (21) that $w_{g}^{(h)} = w_{g}^{(b)}$ if all subsystems in cell $g$ have uniform user distribution over the cell because in this case $w_{g}^{(b)}$ depends only on $\lambda_{c}$ and $d_{g,h}$. Evaluating $w_{g}^{(b)}$ for all neighbour interferent cells from (21) based on the cellular system architecture and geometry, we can determine the total multiple access interference power on receiver $hjl$ from

$$S_{dMl}^{(h)} = \sum_{k} S_{p}^{(e)} w_{g}^{(b)} - S_{g}^{(h)} \quad (23)$$

where $S_{g}^{(h)}$ is the power on cell $g$ received from all users of cell $g$.

A cellular architecture based on hexagonal geometry is described in Fig. 4. All cells have the same radius $\lambda$ and are grouped in rings so that the number of cells in ring-$n$ is $6n$ and the total number of rings is $G$. Base stations are assumed to be at the centre of each cell so that the distance between base stations of cell $h$ and the cell at ring-$n$ and coordinate $m$ is given by $2\lambda \sqrt{n^2 + m^2} - nm$. In this case, the total multiple access interference power $S_{dMl}^{(h)}$ on receiver $hjl$ is given by

$$S_{dMl}^{(h)} = S_{p}^{(e)} \left( 1 + \sum_{n=1}^{G} \sum_{m=1}^{6n} 6w_{g}^{(b)} - S_{g}^{(h)} \right) \quad (24)$$

Figure 4. Hexagonal cellular geometry

IV. BIT ERROR PROBABILITY

The bit detection errors at multi-rate receiver $hjl$ occur when a decision variable $\hat{A}_{hjl}$ is not mapped into the correct decision region $D_{l}$ that corresponds to the transmitted symbol $Z_{y} = Z_{x} + jZ_{y}$. Thus, the bit error probability at receiver $hjl$ is

$$P_{b,hjl} = \sum_{y=0}^{M_{b}} P(A_{hjl} = Z_{x}) \sum_{x=0}^{M_{b}} \log_{2} M_{b} P(A_{hjl} = D_{l}, A_{hjl} = Z_{x}) \quad (25)$$

Figure 3. Interference to neighbour cell
where \( n(x,y) \) is the number of different bits between the \( \log_2 M \)-bit sequences corresponding to symbols \( Z_x \) and \( Z_y \). Consider QPSK modulation \((M_y = 4)\) described in Fig. 5 and assume the difference between adjacent symbols is one bit and the difference between opposite symbols is two bits. Then, for equiprobable symbols, the optimum detection solution is achieved when the decision boundaries \( \mu' \) and \( \mu^0 \) are zero.

\[
\begin{align*}
E \left\{ \hat{A}_{hy} \mid A_{hy} = Z_h \right\} &\quad E \left\{ \hat{A}_{hy} \mid A_{hy} = Z_y \right\} \\
\mu^0 &= 0 \\
E \left\{ \hat{A}_{hy} \mid A_{hy} = Z_y \right\} &\quad E \left\{ \hat{A}_{hy} \mid A_{hy} = Z_x \right\} \\
\mu' &= 0
\end{align*}
\]

Figure 5. Decision regions in QPSK modulation

As \( \eta_{hy} \), \( \eta_{yh} \), \( \gamma_{hy} \) and \( \gamma_{yh} \) are independent Gaussian random variables, decision variables \( \hat{A}_{hy} \) and \( \hat{A}_{yh} \) are also Gaussian random variables and due to symmetry we get from (25) that

\[
P_{b,hy} = Q \left( \frac{\sqrt[2]{\text{Var} \left( \hat{A}_{hy} \right)}}{\sqrt{\text{Var} \left( \hat{A}_{yh} \right)}} \right) \tag{26}
\]

where from (3), (7) and (18) we have

\[
E \left\{ \hat{A}_{hy} \mid A_{hy} = Z_h \right\} = \sqrt{\frac{P_{hy}}{2}} \eta_{hy} \gamma_{hy} T_{h,0} \tag{27}
\]

and

\[
\text{Var} \left( \hat{A}_{hy} \right) = \left( N_0 T_{h,y} / 4 \right) + \left( S_{\text{SNR}}^{(h)y} / 3N_0 \text{SNR}_{hy} / 6 \right). \tag{28}
\]

Hence, the bit error probability at receiver \( h/j \) is given by

\[
P_{b,h/y} = Q \left( \frac{1}{2(S_{\text{SNR}}^{(h)y} / 3N_0 \text{SNR}_{hy})} \right) \tag{29}
\]

where \( (\text{SNR})_{hy} = S_{\text{SNR}}^{(h)y} / R_{hy} N_0 \) is the received signal-to-noise ratio per bit of the signal of user \( h/y \).

V. NUMERICAL RESULTS

In this section we present some results that illustrate bit error probability in multi-rate systems derived in (29).

Consider the multi-rate cellular system proposed on previous sections and assume that all cells of the system have the same power \( S_{\text{SNR}}^{(h)y} \) resulting from its several users who develop various applications. If a new user \( h/y \) is introduced in cell \( h \), his bit error probability \( P_{b,h/y} \) can be evaluated from (29) as a function of his received signal-to-noise ratio per bit \( (\text{SNR})_{hy} \), as shown in Fig. 6. Multiple access interference power \( S_{\text{MAI}}^{(h)y} \) is assumed fixed and given by (24), with \( v_{(h)y} \) given by (21), \( G = 5 \), and the values of \( S_{\text{SNR}}^{(h)y} / N_0 = S_{\text{SNR}}^{(h)y} / N_0 \) are defined in Fig. 6. Observe that an user needs more energy to support a performance requirement when interference is higher.

![Figure 6](image)

VI. CONCLUSIONS

We derived a bit error probability expression for multi-rate DS-WCDMA cellular system, assuming random spreading codes and Gaussian approximation for multiple access interference. A mathematical model that describes interference from neighbour cells as a function of path loss model, user distributions over the cells and cellular architecture was also presented. At last, some results illustrated analytical results obtained. Models and expressions proposed have proved to be very convenient to the investigation of multi-rate systems and can be useful in the study of more complex systems.

REFERENCES


