On the Power Spectral Density of Chaotic Signals Generated by Skew Tent Maps

Daniela Mitie Kato and Marcio Eisencraft
Escola de Engenharia
Universidade Presbiteriana Mackenzie
São Paulo, Brazil
Email: danikato@yahoo.com; marcioft@mackenzie.br

***Abstract*— This paper investigates the characteristics of the Power Spectral Density (PSD) of chaotic orbits generated by skew tent maps. The influence of the Lyapunov exponent on the autocorrelation sequence and on the PSD is evaluated via computational simulations. We conclude that the essential bandwidth of these chaotic signals is strongly related to this exponent and they can be low-pass or high-pass depending on the parameter of the family. These results withstand the usual thought that chaotic signals are always broadband and provide a simple way of generating chaotic sequences with arbitrary bandwidth.**

I. Introduction

A chaotic signal is defined as being deterministic, aperiodic and presenting sensitivity to initial conditions. This last property means that, if the generator system is initialized with a slightly different initial condition, the obtained signal diverges very quickly from the original one [1].

From the Telecommunication Engineering point of view, chaotic signals possess some interesting properties. The literature, e.g. [2], [3], uses to consider that they have broadband, impulsive Autocorrelation Sequence (ACS) and the cross-correlation sequence between orbits with different initial conditions assumes low values. Due to these characteristics, since the beginning of the 1990's, the field of communication with chaotic carriers has received a great deal of attention, see e.g. [2], [4] and references therein. Using chaotic signals to modulate narrowband information signals results in larger bandwidth and lower Power Spectral Density (PSD) level, witch characterize spread spectrum systems [5]. This way, chaotic modulations possess the same qualities than conventional spread spectrum [2], mitigating both multipath and jamming effects.

The study of spectral characteristics is an important issue when it comes to using chaotic signals in practical communications. A great deal of the existing technology is based in frequency multiplexing and, besides, the bandwidth of the transmitted signal is an essential parameter when planning a communication system. Some works as [6]–[10] depict the PSD of continuous-time chaotic signals generated by particular systems. However, the spectral characteristics of these signals have been rarely studied deeply. Most papers just state that they are broadband signals.

The objective of this paper is to present some preliminary results on this far-reaching subject. We investigate via computational simulation the PSD of discrete-time chaotic orbits generated by a family of piecewise linear maps, the skew tent one. Furthermore, we relate a property of the chaotic attractor of these orbits, the *Lyapunov exponent* [1] with a convenient measure of the bandwidth of a signal, the *essential bandwidth* [5].

The paper is organized as follows. Section II presents the skew tent family and its relevant characteristics. The techniques for obtaining the PSD of chaotic signals are presented in Section III. In Section IV the relationship between Lyapunov exponent and essential bandwidth is explored. Finally, we summarize and discuss the key results in Section V.

II. Skew tent maps

A one-dimensional *discrete-time dynamical system* or *map* is defined by the difference equation

$$s(n+1) = f(s(n)), \tag{1}$$

where $f(.)$ is a function with the same domain and range space $U \subset \mathbb{R}$, $n \in \mathbb{N}$ and $s(0) \in U$. For each *initial condition* s_0, an *orbit* or *signal* becomes defined as $s(n, s_0) = f^n(s_0)$ with $f^n(.)$ being the n-th successive application of $f(.)$. For simplicity of notation, an orbit will be symbolized by $s(n)$ whenever s_0 is immaterial.

In this paper, we focus on the skew tent maps defined by

$$s(n+1) = f_I(s(n)) \tag{2}$$

where

$$f_I(s) = \begin{cases} \frac{2}{\alpha+1}s + \frac{1-\alpha}{\alpha+1}, & -1 < s < \alpha \\ \\ \frac{2}{\alpha-1}s - \frac{\alpha+1}{\alpha-1}, & \alpha \leq s < 1 \end{cases} \tag{3}$$

and $\{\alpha, s(0)\} \subset U = (-1, 1)$. This family is a modified version of the one proposed in [11]. The parameter α determines the x-coordinate of the tent's peak. This map is shown in Figure 1(a) along with the orbit $s(n, 0.2)$ for $\alpha = 0.6$ in Figure 1(b).

The Lyapunov exponent h is the divergence rate between nearby orbits and is usually taken as a measure of the "chaoticness" of an aperiodic signal. For the orbit $s(n, s_0)$, it is given by [1]

$$h = \lim_{N\to\infty} \frac{1}{N} \left(\sum_{n=0}^{N-1} \ln |f'(s(n, s_0))| \right). \tag{4}$$

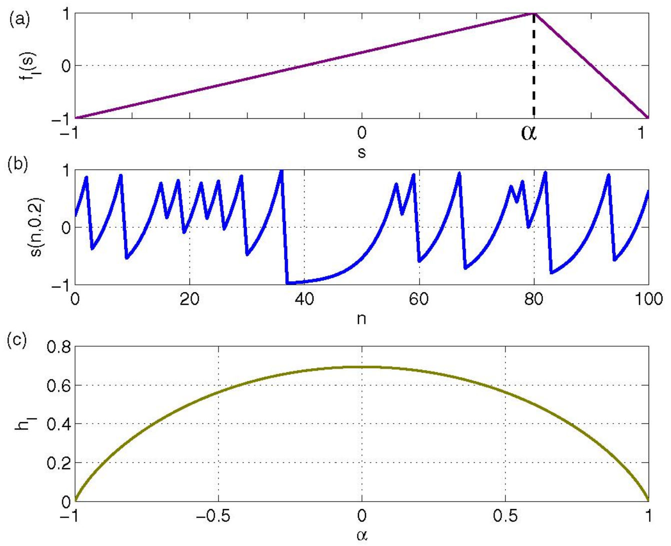

Fig. 1. (a) Skew tent map $f_I(s)$; (b) the orbit $s(n, 0.2)$ for $\alpha = 0.6$ and (c) Lyapunov exponent of the chaotic orbits as a function of α.

A positive h is a sufficient condition for an aperiodic signal to be classified as chaotic. It can be shown [11] that the Lyapunov exponent of almost every orbit of a skew tent map is a function of α only and is given by

$$h_I = \frac{\alpha+1}{2}\ln\left(\frac{2}{\alpha+1}\right) + \frac{1-\alpha}{2}\ln\left(\frac{2}{1-\alpha}\right). \qquad (5)$$

Figure 1(c) shows how h_I varies with α. For every considered value of α, $h_I > 0$ and the maximum value of h_I, $h_{Imax} = \ln 2$, is attained for $\alpha = 0$.

The chaotic orbits generated by Eq. (2) have uniform invariant density over $(-1, 1)$ [12]. Consequently, they are all zero-mean and their average power is $1/3$ independently of α.

In the next two sections we characterize the ACS and the PSD of the signals generated by these maps.

III. PSD of chaotic signals

There are two different ways of interpreting the chaotic signals generated by a given map. They can be seen as deterministic individual signals or as sample-functions of a stochastic process. Each of these interpretation gives rise to different forms of calculating the PSD. Both will be analyzed in this section.

A. *Chaotic signals as deterministic individual signals*

Given the map $f(.)$ in Eq. (1) and the initial condition $s(0) = s_0$, the sequence $s(n, s_0)$ is well defined for all $n \geq 0$ and its ACS can be readily determined as

$$R(l, s_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} s(n, s_0)s(n+l, s_0), \qquad (6)$$

where l is an integer [13]. In this calculation, we consider $s(n+l, s_0) = 0$ whenever $n + l$ results in a negative number.

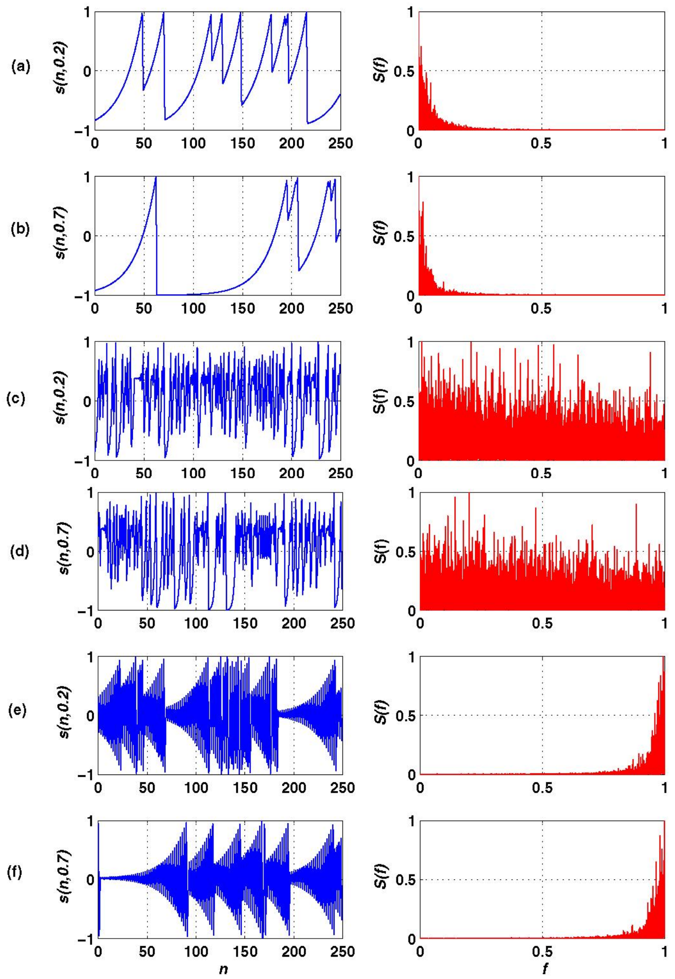

Fig. 2. PSD of individual orbits $s(n, s_0)$: (a) $s_0 = 0.2$, $\alpha = 0.9$; (b) $s_0 = 0.7$, $\alpha = 0.9$; (c) $s_0 = 0.2$, $\alpha = 0.1$; (d) $s_0 = 0.7$, $\alpha = 0.1$; (e) $s_0 = 0.2$, $\alpha = -0.9$; (f) $s_0 = 0.7$, $\alpha = -0.9$

The PSD $S(f, s_0)$ is the Discrete-Time Fourier Transform (DTFT) of $R(l, s_0)$, considering l as the time variable [13]:

$$S(f, s_0) = \sum_{l=-\infty}^{\infty} R(l, s_0)\, e^{-j\pi f l}. \qquad (7)$$

Figure 2 shows six different orbits and their estimated PSD using $N = 20000$ samples. For the PSD plots, the horizontal scale is the normalized frequency f. This way, $f = 1$ is equivalent to the discrete-time frequency $\omega = \pi$ rad/samples and to the continuous-time frequency $f_c = f \cdot f_s/2$, where f_s is the sampling frequency. The PSD curves were normalized so that their maximum value is 1.

Based on these computational simulations we can state that:

i) when the parameter α is positive, the generated signals varies slowly in time and they are low-pass signals, as can be seeing in Figure 2(a) and (b);
ii) when $|\alpha|$ is next to zero, the generated signals are broadband, as in Figure 2(c) and (d);
iii) for negative values of α, the orbits oscillate quickly in time and they are high-pass signals, as in Figure 2(e)

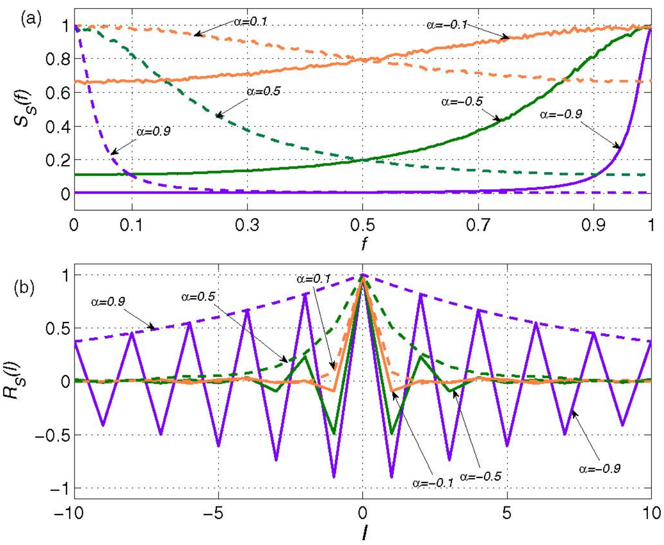

Fig. 3. (a) PSD and (b) ACS of the ensemble of orbits defined by different values of the parameter α. The ACS is normalized so that $R_S(0) = 1$.

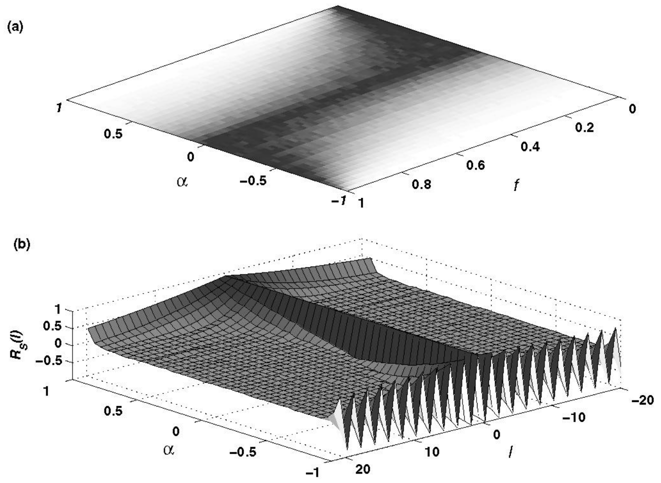

Fig. 4. (a) PSD and (b) ACS of orbits of skew tent maps as a function of α. The darker the point, the higher the associated value.

and (f);

iv) orbits generated by the same map with different initial conditions present similar PSD despite the fact that they are pointwise different in time. This can be concluded from the comparison of Figure 2(a) and (b), (c) and (d) or (e) and (f).

This way, the map, defined by α, is determinant in the spectral characteristics of the signals it generates. The spectral similarities between orbits generated by the same map motivates the interpretation of a chaotic signal as a realization of a stochastic process.

B. Chaotic signals as sample functions of a stochastic process

Chaotic signals generated by a fixed map can be understood as a stochastic process in which each initial condition defines a sample function [12]. This interpretation has the advantage of highlighting properties that apply to the entire set of chaotic orbits defined by the map.

In this case, the map defines an ergodic process [12] and we can define the ACS as

$$R_S(l) = E\left[R\left(l, s_0\right)\right], \tag{8}$$

where the expectation is taken over all initial conditions that generate chaotic orbits. The PSD $S_S(f)$ is the DTFT of $R_S(l)$, in the same way it is done with conventional stochastic processes [14].

Figure 3 shows estimates of the PSD and of the normalized ACS for different values of α. For each curve, the expectation in (8) was estimated considering 20000 orbits with $N = 440$ samples and initial conditions s_0 uniformly distributed in U. The evolution of the PSD and ACS for increasing values of α are plotted in Figure 4.

These figures suggest that:

i) the higher the absolute value of α, the narrower the bandwidth of the generated chaotic signals;

ii) the signal of α defines if the obtained signals are low-pass or high-pass;

iii) the PSDs of the signals generated by α and $-\alpha$ present symmetry around $f = 0.5$, as can be seen in Figure 3(a);

iv) for $\alpha > 0$, $R(l)$ is monotonically decreasing with $|l|$. For $\alpha < 0$, $R(l)$ oscillates indicating that in this case for almost any n and s_0, the signals of $s\left(n, s_0\right)$ and $s\left(n+1, s_0\right)$ are different;

v) it is worth to note that for $\alpha = 0$, the map $f_I(.)$ coincides with the one used in [15] for $\beta = 2$. In this situation, that paper has demonstrated that the generated signals have white spectrum. Our results agree perfectly with theirs;

vi) changing α, it is possible to obtain low-pass or high-pass chaotic signals with arbitrary bandwidth.

These results mean that chaos is far way from being a synonym for broadband non-correlated signals. This way, when it comes to employ chaotic signals in communication systems, it is relevant to investigate their spectral characteristics.

IV. Essential bandwidth and the Lyapunov exponent

The bandlimiting properties of signals can be measured by the *essential bandwidth* defined as the frequency range where 95% of the total signal power is concentrated [5]. We use here a normalized version of this definition, $0 \leq B \leq 1$, dividing the essential bandwidth by 0.95. Using this definition, a white noise has $B = 1$.

From the curves in Figure 3(a), we see that the value of B is determined by the absolute value of α. This control is justified by the direct relationship between this parameter and the Lyapunov exponent shown in Figure 1(c). The lower the absolute value of α, the higher the value of h_I, which means that the orbits diverge faster from nearby ones and the

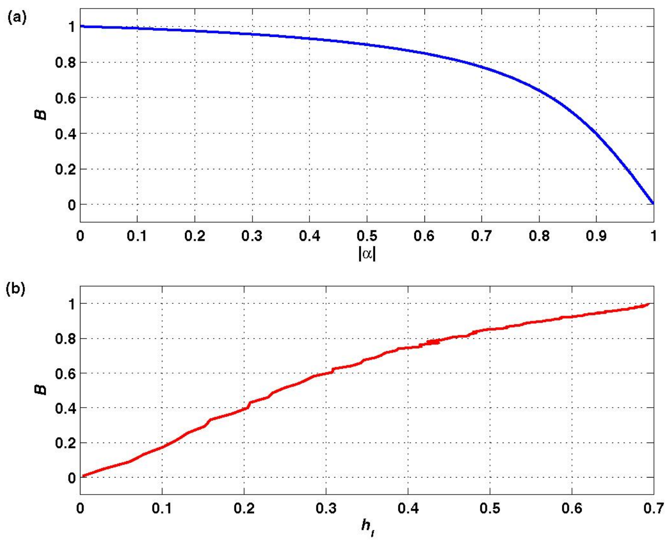

Fig. 5. Essential bandwidth B as a function of (a) $|\alpha|$ and of (b) the Lyapunov exponent h_I.

ACS tends to an impulsive format. Consequently, the PSD, being the DTFT of the ACS, will have a larger B. Figure 5 presents curves of B as function of $|\alpha|$ and of h_I. One can see that we go from a white spectrum at $\alpha = 0$ to an extremely narrowband signal at $|\alpha|$ next to the unity.

The existence of a one-to-one relationship between B and $|\alpha|$ is significative. Choosing a convenient α, it is possible to generate a low-pass or high-pass chaotic signal with arbitrary essential bandwith. Besides, it is possible to use the PSD of an observed orbit to estimate α.

This relationship can also be used to implement new ideas for receivers in chaotic digital modulation systems. An alternative would be to associate different symbols with different values of $|\alpha|$ and to transmit N points of an orbit generated by the corresponding map. In the receiver the essential bandwidth of the received signal would be estimated and $|\alpha|$ together with the associated symbol would be determined.

The performance of the above possibilities are subjects of further research.

V. Conclusions

This paper analyzes through computational simulations the PSD of the orbits generated by the skew tent maps (2). We have shown the influence of the parameter α and of the Lyapunov exponent on the ACS and PSD of the obtained chaotic signals.

An important preliminar consequence of our results is that there are situations when the chaotic signals generated by one-dimensional maps are not broadband. Furthermore, the ACS of these signals is not necessarily impulsive. Citing the opposite as advantages of using chaotic signals need more careful analysis. It is possible to generate low-pass or high-pass chaotic signals with arbitrary B very easily.

The strong relationship between Lyapunov exponent and bandwidth can be useful in chaotic estimation and modulation problems, as discussed in Section IV. Following this path, there are many possibilities to explore in future researches.

The generalization of our results to other one-dimensional maps seems to be possible using the *conjugacy* concept [1]. Numerical simulations show that conjugated maps generate orbits with similar spectral characteristics. This subject is under research.

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