Conditions for Synchronizing a Master-Slave Bandlimited Chaos-based Communication System

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***Abstract*— In recent years, many communication systems that encode information in a chaotic signal were studied. To tackle bandwidth constraints, bandlimited chaos-based communication systems were proposed, showing the possibility of controlling the chaotic signals spectra using digital filters. Since these filters modify the original system, it is necessary to study how their insertion affects the chaotic synchronization. In this work we demonstrate that synchronization is attained independently of the filters coefficients.**

***Index Terms*— Chaos-based communication; bandlimited channels; synchronization.**

I. Introduction

In Telecommunication and Signal Processing areas, the researches involving chaotic signals have rised after Pecora and Carrol seminal work [1] showing that two identical systems, generating chaotic signals, could be synchronized.

Some interesting and innovative communication schemes based on chaotic synchronization were proposed [2]–[7]. However, they seldom surpassed the frontier between theoretical and laboratory setup to practical or commercial environments. One important reason for this fact is the sensibility of the chaotic synchronization to channel imperfections, like noise or distortion [4], [6], [8], [9].

A simple way to synchronize master-slave chaotic systems was proposed by Wu and Chua [10]. Their scheme is based in the separation of linear and non-linear components of the involved systems. As long as the linear part is stable and the non-linear component is transmitted from master to slave, both systems will completely synchronize [11].

A discrete-time chaos-based communication system (CBCS) based on Wu and Chua synchronization, was proposed in [12]. Since chaotic signals have large bandwidth and the physical transmission channels are always bandlimited, it is also addressed in [12] the bandwidth controlling issue, showing a CBCS capable of generating and transmitting bandlimited signals by placing digital filters in the feedback loops of the master and the slave systems. The synchronization conditions for this bandlimited system was analytically determined for the Hénon map in [13]. In [4] the performance of this CBCS was addressed in terms of bit error rate for an additive white Gaussian noise channel and a tuning scheme to filter the out-of-band noise was proposed.

In this work, we extend the results of [13] presented in [4] by analytically determining the chaotic synchronization conditions, in terms of the filters coefficients, for any chaotic generator map.

The paper is organized as following: in Section II we review the bandlimited CBCS described in [12]. Next, in Section III, the analytic conditions for chaotic synchronization is derived. Finally, in Section IV, we draft some conclusions.

II. A Bandlimited CBCS

Consider a master-slave discrete-time CBCS described by [12]

$$\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n) + \mathbf{b} + \mathbf{f}(s(n)) \quad (1)$$

$$\mathbf{y}(n+1) = \mathbf{A}\mathbf{y}(n) + \mathbf{b} + \mathbf{f}(r(n)) \quad (2)$$

where $n \in \mathbb{N}$ represents time instants, $\mathbf{A}_{K\times K}$ and $\mathbf{b}_{K\times 1}$ are constants, $\{\mathbf{x}(n), \mathbf{y}(n)\} \subset \mathbb{R}^K$, $\mathbf{x}(n) = [x_1(n), x_2(n), \ldots, x_K(n)]^T$ and $\mathbf{y}(n) = [y_1(n), y_2(n), \ldots, y_K(n)]^T$. The function $\mathbf{f}(\cdot) : \mathbb{R}^K \to \mathbb{R}^K$ is non-linear in general, $s(n)$ is the transmitted signal and $r(n)$ the received signal. We consider a bandlimited noiseless channel represented by a finite impulse response (FIR) $h_c(n)$. This way,

$$r(n) = s(n) * h_c(n) = \sum_{k=0}^{N_c-1} s(k)h_c(n-k), \quad (3)$$

where "$*$" is the convolution operator. This CBCS is ilustrated by the diagram in Figure 1 considering $H_S(\omega) = 1$, so that $x_{k+1}(n) = x_1(n)$ and $y_{k+1}(n) = y_1(n)$.

Using (1)-(2), the synchronization error, $\mathbf{e}(n) \triangleq \mathbf{x}(n) - \mathbf{y}(n)$, can be written as $\mathbf{e}(n+1) = \mathbf{A}\mathbf{e}(n)$, so master and slave systems completely synchronize if the eigenvalues λ_i of $\mathbf{A}$ satisfy [14]

$$|\lambda_i| < 1, \; 1 \le i \le K. \quad (4)$$

Since the eigenvalues λ_i, from matrix $\mathbf{A}$, determine whether the systems (1) and (2) synchronize, $\mathbf{A}$ is called *synchronization matrix*.

The message or information, $m(n)$, is encoded by the chaotic signal $x_1(n)$ through the invertible function $c(\cdot,\cdot)$,

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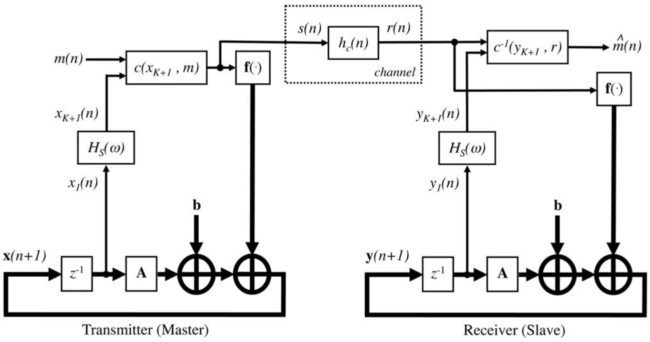

Fig. 1. A Bandlimited CBCS [12].

resulting in the transmitted signal $s(n) = c(x_1(n), m(n))$. This way, $m(n)$ can be decoded by

$$m(n) = c^{-1}(x_1(n), s(n)). \tag{5}$$

If the eigenvalues of $\mathbf{A}$ satisfies (4), $y_1(n) \to x_1(n)$ and if we define the recovered message as $\hat{m}(n) = c^{-1}(y_1(n), r(n))$, using (5), $\hat{m}(n) \to m(n)$. Clearly this may not be the case for a nonideal channel where $r(n) \neq s(n)$ and consequently $\hat{m}(n) \neq m(n)$. The Hénon map [15] is used as the chaotic generator as in [4], [13].

In Figure 2 we show typical signals $m(n)$, $s(n)$ and $\hat{m}(n)$ in time and frequency domains for an ideal channel and $s(n) = c(x_1(n), m(n)) = 0.9x_1(n) + 0.1m(n)$. It is visible that $m(n)$ is retrieved without error in the receiver. However, as showed in Figure 3, if $h_c(n)$ is a low-pass FIR filter with cut-off frequency $\omega_c = 0.95\pi$ and order $N_c = 200$, the signal $\hat{m}(n)$ is completely different from $m(n)$, showing that the channel bandwidth, even in an ideal condition in terms of noise, corrupts the chaotic synchronization. As the receiver is non-linear, any modified spectrum component received can affect all the other components of the retrieved signal.

Since chaotic signals are broadband in general, $s(n)$ will be broadband. One solution for reestablishing the chaotic synchronization through bandlimited channels is adjusting the spectrum of $s(n)$ using FIR filters, $H_S(\omega)$, in the feedback loops [12]. The spectrum of $x_1(n)$ is limited using a low-pass filter with cut-off frequency ω_S. This way, for each input $x_1(n)$, the output $x_{K+1}(n)$ is written as

$$x_{K+1}(n) = \sum_{j=0}^{N_S-1} c_j x_1(n-j) \tag{6}$$

where c_j, $0 \leq j \leq N_S - 1$, are the $H_S(\omega)$ filter coefficients. Choosing $H_S(\omega)$, $m(n)$ and the coding function adequately the signal $s(n)$ will be essentially bandlimited. In [13] it was shown that for the Hénon map, the synchronization is not affected by the filter coefficients. This means that for any FIR filter $H_S(\omega)$, only the original synchronization matrix is relevant in determining whether master-slave synchronization is maintained.

In Figure 4 we show the signals $m(n)$, $s(n)$ and $\hat{m}(n)$ for the same conditions described for Figure 3, using filters $H_S(\omega)$

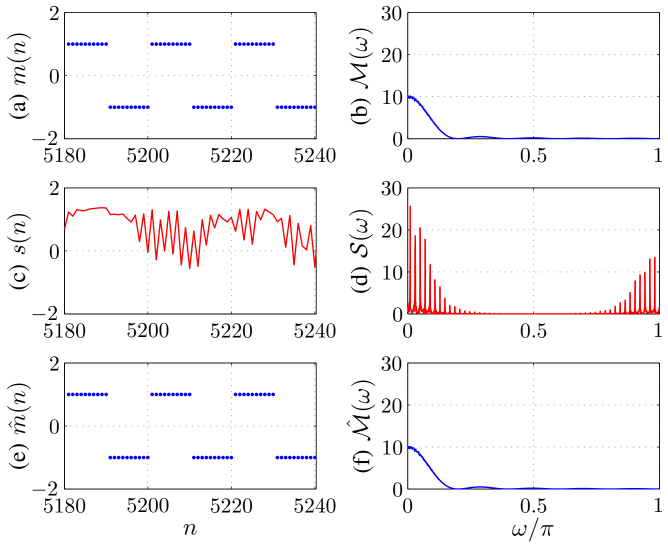

Fig. 2. Examples of $m(n)$, $s(n)$ and $\hat{m}(n)$ in time and frequency domains for the CBCS of Figure 1 with $H_S(\omega) = 1$ under an ideal channel.

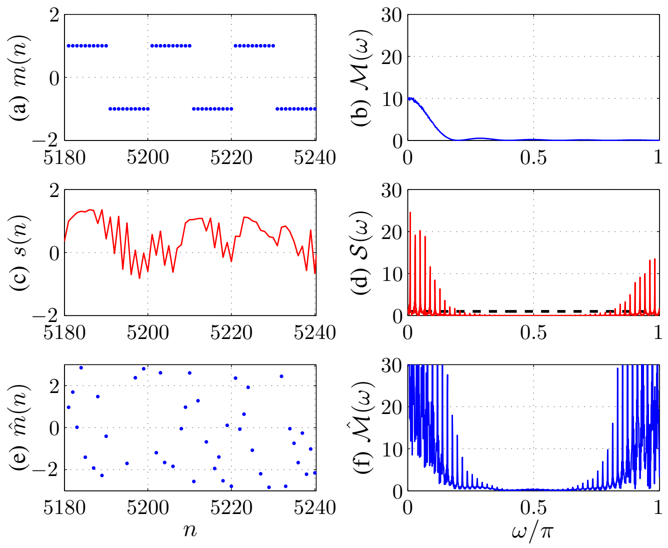

Fig. 3. Examples of $m(n)$, $s(n)$ and $\hat{m}(n)$ in time and frequency domains for the CBCS of Figure 1 with $H_S(\omega) = 1$ under a low-pass channel with $\omega_c = 0.95\pi$. The channel frequency response is shown in dashed line in (d).

in master-slave systems with cut-off frequency $\omega_S = 0.4\pi$ and order $N_S = 200$. Clearly, the message $m(n)$ is retrieved without error showing that chaotic synchronization can be obtained in bandlimited chaos-based communication system.

In Section III, we extend this result for any K-dimensional map.

III. Synchronization Conditions

Being a_{ij}, $1 \leq i, j \leq K$, the synchronization matrix $\mathbf{A}$ coefficients and c_j, $0 \leq j \leq N_S - 1$, the $H_S(\omega)$ filter coefficients, the state equations that describe the Figure 1

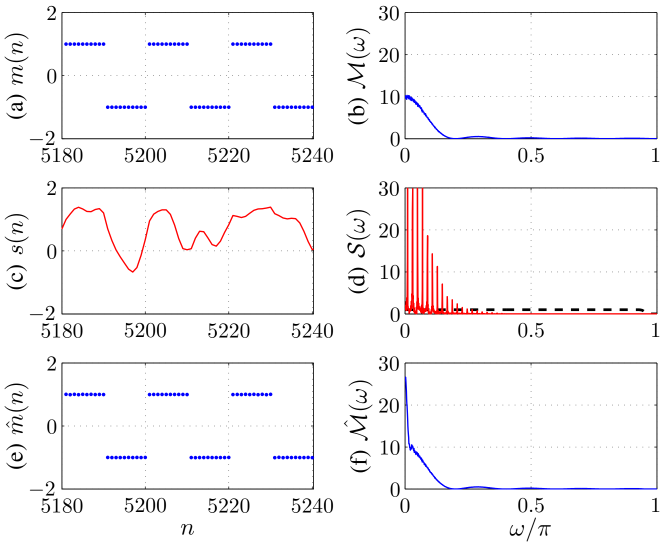

Fig. 4. Examples of $m(n)$, $s(n)$ and $\hat{m}(n)$ in time and frequency domain for the CBCS of Figure 1 with $\omega_S = 0.4\pi$ under a low-pass channel with $\omega_c = 0.95\pi$. The channel frequency response is shown in dashed line in (d).

master system are

$$\begin{cases} x_1(n+1) &= a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1K}x_K(n) + b_1 + f(s(n)) \\ x_2(n+1) &= a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2K}x_K(n) + b_2 \\ \vdots & \\ x_K(n+1) &= a_{K1}x_1(n) + a_{K2}x_2(n) + \cdots + a_{KK}x_K(n) + b_K \\ x_{K+1}(n+1) &= c_0x_1(n+1) + c_1x_1(n) + \cdots + c_{N_S-1}x_1(n-N_S+2) \end{cases}, \tag{7}$$

with $s(n) = c\left(x_{K+1}(n), m(n)\right)$.

For $N_S \geq 4$, defining

$$\begin{aligned} x_{K+2}(n) &= x_1(n-1) \\ x_{K+3}(n) &= x_{K+2}(n-1) \\ &\vdots \\ x_{K+N_S-1}(n) &= x_{K+N_S-2}(n-1), \end{aligned} \tag{8}$$

(7) can rewriting as

$$\begin{cases} x_1(n+1) &= a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1K}x_K(n) + b_1 + f(s(n)) \\ x_2(n+1) &= a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2K}x_K(n) + b_2 \\ \vdots & \\ x_K(n+1) &= a_{K1}x_1(n) + a_{K2}x_2(n) + \cdots + a_{KK}x_K(n) + b_K \\ x_{K+1}(n+1) &= (c_0a_{11}+c_1)x_1(n) + c_0a_{12}x_2(n) + \cdots + c_0a_{1K}x_K(n) + c_2x_{K+2}(n) + \cdots \\ & \quad + c_{N_S-1}x_{K+N_S-1}(n) + c_0b_1 + c_0f(s(n)) \\ x_{K+2}(n+1) &= x_1(n) \\ x_{K+3}(n+1) &= x_{K+2}(n) \\ \vdots & \\ x_{K+N_S-1}(n+1) &= x_{K+N_S-2}(n) \end{cases}. \tag{9}$$

The insertion of $H_S(\omega)$ modify the master system order to $K' = K + N_S - 1$. This system can be written again as (1), with a synchronization matrix $\mathbf{A}'$.

For $N_S \geq 4$ this matrix $\mathbf{A}'$ with dimension $K' \times K'$, is described as

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A} & & \mathbf{O}_{K\text{x}(N_S-1)} \\ & \mathbf{c}_{1\text{x}K'} & \\ & \mathbf{u}_{1\text{x}K'} & \\ \mathbf{O}_{(N_S-3)\text{x}(K+1)} & \mathbf{I}_{(N_S-3)\text{x}(N_S-3)} & \mathbf{O}_{(N_S-3)\text{x}1} \end{bmatrix}, \tag{10}$$

where $\mathbf{O}$ is a null matrix, $\mathbf{I}$ is the identity matrix, $\mathbf{c} = [(c_0a_{11} + c_1), c_0a_{12}, \ldots, c_0a_{1K}, 0, c_2, \ldots, c_{N_S-1}]$ and $\mathbf{u} = [1, 0, 0, \ldots, 0]$.

Particularly for $N_S = 1$,

$$\begin{cases} x_1(n+1) &= a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1K}x_K(n) + b_1 + f(s(n)) \\ x_2(n+1) &= a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2K}x_K(n) + b_2 \\ \vdots & \\ x_K(n+1) &= a_{K1}x_1(n) + a_{K2}x_2(n) + \cdots + a_{KK}x_K(n) + b_K \\ x_{K+1}(n+1) &= c_0a_{11}x_1(n) + c_0a_{12}x_2(n) + \cdots + c_0a_{1K}x_K(n) + c_0b_1 + c_0f(s(n)) \end{cases}. \tag{11}$$

and

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A} & \mathbf{O}_{K\times 1} \\ \mathbf{c}_{1\times(K+1)} & \end{bmatrix}, \tag{12}$$

with $\mathbf{c} = [c_0a_{11}, c_0a_{12}, \ldots, c_0a_{1K}, 0]$.

For $N_S = 2$

$$\begin{cases} x_1(n+1) &= a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1K}x_K(n) + b_1 + f(s(n)) \\ x_2(n+1) &= a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2K}x_K(n) + b_2 \\ \vdots & \\ x_K(n+1) &= a_{K1}x_1(n) + a_{K2}x_2(n) + \cdots + a_{KK}x_K(n) + b_K \\ x_{K+1}(n+1) &= (c_0a_{11}+c_1)x_1(n) + c_0a_{12}x_2(n) + \cdots + c_0a_{1K}x_K(n) + c_0b_1 + c_0f(s(n)) \end{cases}. \tag{13}$$

and

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A} & \mathbf{O}_{K\times 1} \\ \mathbf{c}_{1\times(K+1)} & \end{bmatrix}, \tag{14}$$

with $\mathbf{c} = [(c_0a_{11} + c_1), c_0a_{12}, \ldots, c_0a_{1K}, 0]$.

Finally, for $N_S = 3$

$$\begin{cases} x_1(n+1) &= a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1K}x_K(n) + b_1 + f(s(n)) \\ x_2(n+1) &= a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2K}x_K(n) + b_2 \\ \vdots & \\ x_K(n+1) &= a_{K1}x_1(n) + a_{K2}x_2(n) + \cdots + a_{KK}x_K(n) + b_K \\ x_{K+1}(n+1) &= (c_0a_{11}+c_1)x_1(n) + c_0a_{12}x_2(n) + \cdots + c_0a_{1K}x_K(n) + c_2x_{K+2}(n) + \cdots \\ & \quad + c_{N_S-1}x_{K+N_S-1}(n) + c_0b_1 + c_0f(s(n)) \\ x_{K+2}(n+1) &= x_1(n) \end{cases} \tag{15}$$

and

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A} & \mathbf{O}_{K\times(N_S-1)} \\ \mathbf{c}_{1\times K'} & \\ \mathbf{u}_{1\times K'} & \end{bmatrix}, \tag{16}$$

with $\mathbf{c} = [(c_0a_{11} + c_1), c_0a_{12}, \ldots, c_0a_{1K}, 0, c_2]$ e $\mathbf{u} = [1, 0, 0, \ldots, 0]$.

The following theorem establishes the relationship between matrix $\mathbf{A}$ and $\mathbf{A}'$ eigenvalues.

Theorem 3.1: The matrix $\mathbf{A}'$ has K eigenvalues identical to the ones of matrix $\mathbf{A}$ and $(K' - K)$ null eigenvalues.

Proof: The eigenvalues λ_i, $1 \leq i \leq K$, of $\mathbf{A}$, are the roots of

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (\lambda - a_{11}) & a_{12} & \cdots & a_{1K} \\ a_{21} & (\lambda - a_{22}) & \cdots & a_{2K} \\ \vdots & \vdots & (\lambda - \cdots) & \vdots \\ a_{K1} & a_{K2} & \cdots & (\lambda - a_{KK}) \end{vmatrix} = 0. \tag{17}$$

The matrix $\mathbf{A}'$ is generically described as

$$\mathbf{A}' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1K} & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & a_{2K} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{K1} & a_{K2} & \cdots & a_{KK} & 0 & 0 & \cdots & 0 & 0 \\ (c_0a_{11}+c_1) & c_0a_{12} & \cdots & c_0a_{1K} & 0 & c_2 & \cdots & c_{N_S-2} & c_{N_S-1} \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{18}$$

and the eigenvalues λ_i', $1 \leq i \leq K'$, of $\mathbf{A}'$, the roots of

$$\det(\lambda'\mathbf{I}-\mathbf{A}') = \begin{vmatrix} (\lambda'-a_{11}) & a_{12} & \cdots & a_{1K} & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & (\lambda'-a_{22}) & \cdots & a_{2K} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{K1} & a_{K2} & \cdots & (\lambda'-a_{KK}) & 0 & 0 & \cdots & 0 & 0 \\ (c_0a_{11}+c_1) & c_0a_{12} & \cdots & c_0a_{1K} & \textcircled{\lambda'} & c_2 & \cdots & c_{N_S-2} & c_{N_S-1} \\ 1 & 0 & \cdots & 0 & 0 & \lambda' & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \lambda' \end{vmatrix} = 0. \tag{19}$$

Applying the Laplace Expansion Theorem (LET), choosing the element $a_{(K+1)(K+1)} = \lambda'$, circled in (19), results

$$\det(\lambda'\mathbf{I}-\mathbf{A}') = \lambda' \begin{vmatrix} (\lambda'-a_{11}) & a_{12} & \cdots & a_{1K} & 0 & \cdots & 0 & 0 \\ a_{21} & (\lambda'-a_{22}) & \cdots & a_{2K} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{K1} & a_{K2} & \cdots & (\lambda'-a_{KK}) & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & \lambda' & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & \textcircled{\lambda'} \end{vmatrix} = 0. \tag{20}$$

Successively applying the LET $(N_S - 1)$ times, we obtain

$$\det(\lambda'\mathbf{I}-\mathbf{A}') = \lambda'^{(N_S-1)} \begin{vmatrix} (\lambda'-a_{11}) & a_{12} & \cdots & a_{1K} \\ a_{21} & (\lambda'-a_{22}) & \cdots & a_{2K} \\ \vdots & \vdots & (\lambda'-\cdots) & \vdots \\ a_{K1} & a_{K2} & \cdots & (\lambda'-a_{KK}) \end{vmatrix} = 0. \tag{21}$$

Comparing (21) to (17)

$$\det(\lambda'\mathbf{I}-\mathbf{A}^{'}) = \lambda^{'(N_S-1)}\det(\lambda\mathbf{I}-\mathbf{A}) = 0. \tag{22}$$

The solutions of (17) are

$$\begin{cases} \lambda_1' & = & \lambda_1 \\ \lambda_2' & = & \lambda_2 \\ \vdots & = & \vdots \\ \lambda_K' & = & \lambda_K \\ \lambda_{K+1}' & = & 0 \\ \vdots & = & \vdots \\ \lambda_{K+N_S-1}' & = & 0 \end{cases} \tag{23}$$

and the theorem is proved. ■

This way, if the original master-slave system synchronize, i.e., the eigenvalues of $\mathbf{A}$ satisfy (4), the introduction of the filter $H_S(\omega)$ does not affect the synchronization, since the eigenvalues of $\mathbf{A}'$ also satisfy (4).

This result generalize the previous one obtained in [13] for Hénon map, opening the possibility of finding a map that can improve BER performance in CBCS.

IV. Conclusions

In this work we present an extended analysis of a bandlimited chaos-based communication system. We determine, analytically, the necessary conditions for master-slave chaotic synchronization in terms of the filters coefficients, for any K-dimensional chaos generator map. How to analytically access the effect of FIR filters on chaotic nature of the transmitted signals are under research.

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