Chaotic Synchronization in Discrete-Time Systems Connected by Bandlimited Channels

M. Eisencraft, R. D. Fanganiello, and L. H. A. Monteiro

***Abstract*—Due to the broadband characteristic of chaotic signals, many of the methods that have been proposed for synchronizing chaotic systems do not usually present a satisfactory performance when applied to bandlimited communication channels. Here, the effects of bandwidth limitations imposed by the channel on the synchronous solution of a discrete-time chaotic master-slave network are investigated. The discrete-time system considered in this study is the Hénon map. It is analytically shown that synchronism can be achieved in such a network by introducing a digital filter in the feedback loop responsible for generating the chaotic signal that will be sent to the slave node. Numerical simulations relating the filter parameters, such as its order and cut-off frequency, to the maximum Lyapunov exponent of the master node, which determines if the transmitted signal is chaotic or not, are also presented. These results can be useful for practical communication schemes based on chaos.**

***Index Terms*—Bandlimited channel, chaos, Hénon map, master-slave network, synchronization.**

I. Introduction

In telecommunications, synchronism of periodic oscillators composing the network can be fundamental for correctly accomplishing information processing tasks (e.g. [1]). Two well-known strategies developed for synchronizing regular clocks are SDH (Synchronous Digital Hierarchy) and SONET (Synchronous Optical Network) (e.g. [2]), which motivated many theoretical studies (e.g. [3], [4]). In these architectures, synchronism means that all nodes constituting the network oscillate in the same frequency.

In the last two decades, the feasibility of communication systems based on the synchronism of chaotic systems has been theoretically and experimentally investigated (e.g. [5], [6]). Chaotic synchronization means coincidence of the states of the connected systems. A chaotic system deterministically generates trajectories in the state space that are aperiodic, limited and extremely sensitive to the initial condition (e.g. [7]). This sensitivity is evaluated by the Lyapunov exponent. The existence of a positive exponent implies chaos (e.g. [7]). Chaotic signal is conjectured to be used in communication schemes because of its inherent wideband characteristic; a feature necessary, for example, for communication employing spread spectrum techniques (e.g. [8]).

There are several schemes for accomplishing chaotic synchronization (e.g. [9], [10]). For instance, in *chaotic masking* (e.g. [11]), the message to be sent to the slave is simply added to a particular state variable of the master before being transmitted. This state variable concerning the master is a chaotic signal much stronger than the original message. In *chaotic modulation* (e.g. [12]), the message is combined to a particular state variable of the master by applying a coding function, in order to modify the master dynamics. This chaotic signal is then transmitted and the message is recovered in the slave by using a decoding function.

Chaotic signal is usually wideband (e.g. [8], [13], [14]), which becomes a problem for synchronization when the communication channel imposes bandwidth limitations [15]. Due to the nonlinear nature of the network nodes, the spectrum of the signal originally generated by the master can be strongly altered and the message can not be faithfully recoverable in the slave if any spectral component is damaged in the transmission. In fact, even minute gain or phase changes are enough to hamper synchronization. A possible solution to this problem, independently proposed in [16] and [17], demands the use of identical filters in the master and in the slave, in order to confine the spectral content of the transmitted signal to the available bandwidth. Both these works deal with continuous-time systems. A study regarding discrete-time chaotic systems was already numerically performed in [18]. This study was motivated by the ease of employing digital signal processors or microcontrollers for a real implementation. However, in continuous or discrete-time, the requirements on the filter coefficients for preserving the chaotic nature of the transmitted signal and maintain synchronization were not yet determined.

Here, the conditions for synchronizing discrete-time chaotic systems connected by bandlimited channels are analytically derived. The discrete-time system considered in this analysis is the well-known Hénon map (e.g. [5], [19]). This paper is organized as follows. In Section 2, the synchronization method is analytically investigated. In Section 3, the relation among the filter parameters and the maximum Lyapunov exponent of the master is numerically explored. In Section 4, the main conclusions and the possible implications for chaos-based communication schemes are stressed.

II. Discrete-Time Chaos Synchronization in Bandlimited Channels

Consider two discrete-time systems defined by

$$\mathbf{x}(n+1) = A\mathbf{x}(n)+\mathbf{b}+\mathbf{f}(\mathbf{x}(n)) \quad (1)$$

$$\mathbf{y}(n+1) = A\mathbf{y}(n)+\mathbf{b}+\mathbf{f}(\mathbf{x}(n)), \quad (2)$$

where $n \in \mathbb{N}$, $\mathbf{x}(n) = [x_1(n), x_2(n), \ldots, x_k(n)]^T$, $\mathbf{y}(n) = [y_1(n), y_2(n), \ldots, y_k(n)]^T$. The square matrix $A_{k\times k}$ and the column vector $\mathbf{b}_{k\times 1}$ are constants. The function $\mathbf{f} : \mathbb{R}^k \mapsto \mathbb{R}^k$ is nonlinear. The system described by Eq. (1) is autonomous

Manuscript received December 21, 2010. The associate editor coordinating the review of this letter and approving it for publication was Y.-C. Wu.

M. Eisencraft (corresponding author) is with Centro de Engenharia, Modelagem e Ciências Sociais Aplicadas, Universidade Federal do ABC (e-mail: marcio.eisencraft@ufabc.edu.br).

R. D. Fanganiello is with Escola de Engenharia, Universidade Presbiteriana Mackenzie.

L. H. A. Monteiro is with Escola de Engenharia, Universidade Presbiteriana Mackenzie, and with Escola Politécnica, Universidade de São Paulo.

Digital Object Identifier 10.1109/LCOMM.2011.040111.102309

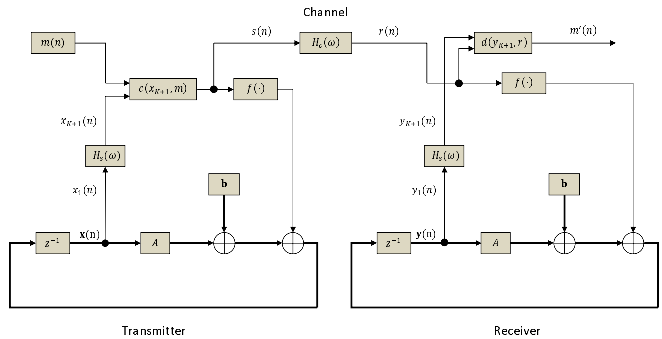

Fig. 1. Discrete-time communication system for bandlimited channels.

and hence is called *master*; the system described by Eq. (2) depends on $\mathbf{x}(n)$ and is called *slave*. The synchronization error $\mathbf{e}(n) \equiv \mathbf{y}(n) - \mathbf{x}(n)$ is given by

$$\mathbf{e}(n+1) = A\mathbf{e}(n). \tag{3}$$

Both systems are *completely synchronized* if $\mathbf{e}(n) \to \mathbf{0}$ as n grows. This occurs if the eigenvalues λ_i of A satisfy (e.g. [20])

$$|\lambda_i| < 1, \quad 1 \le i \le k. \tag{4}$$

Therefore, for a node described by Eq. (1) and obeying the condition (4), it is easy to set up a slave able of synchronizing with it [12].

A chaotic modulation scheme based on the method above was proposed in [18]. It considers that $\mathbf{f}$ depends solely on the component x_1 of $\mathbf{x}$; thus $\mathbf{f}(\mathbf{x}(n)) = [f(x_1(n)), 0, \ldots, 0]^T$. The communication channel is represented by a linear system with frequency response $H_c(\omega)$. In such a scheme, the transmitted signal s is generated by coding the message $m(n)$ by using $x_1(n)$ via coding function c; that is $s(n) = c(x_1(n), m(n))$. Let $r(n) \equiv s(n) * h_c(n)$ be the signal that the channel delivers to the slave, where $h_c(n)$ is the impulse response of the channel and "$*$" represents linear convolution. The message $m'(n)$ decoded in the slave by using the inverse decoding function $d = c^{-1}$ is obtained by $m'(n) = d(y_1(n), r(n))$, where y_1 is the first component of $\mathbf{y}$. Of course, the goal is $m'(n) = m(n)$.

In this synchronization scheme, the two nodes are ruled by

$$\mathbf{x}(n+1) = A\mathbf{x}(n) + \mathbf{b} + \mathbf{f}(s(n)) \tag{5}$$
$$\mathbf{y}(n+1) = A\mathbf{y}(n) + \mathbf{b} + \mathbf{f}(r(n)). \tag{6}$$

Notice that the differences between Eqs. (1,2) and Eqs. (5,6) are only in the arguments of $\mathbf{f}$.

For an ideal channel, i.e., $H_c(\omega) = 1$ and $s(n) = r(n)$, the error dynamics is still governed by Eq. (3). If the condition (4) holds, then $\mathbf{y}(n) \to \mathbf{x}(n)$ and, in particular, $y_1(n) \to x_1(n)$. Hence $m'(n) \to d(x_1(n), c(x_1(n), m(n))) = m(n)$. Therefore, when the parameter values of the master and the slave are perfectly matched over an ideal channel, the message is faithfully recovered at the slave without degradation (except during a synchronization transient).

For a non-ideal channel, i.e., $H_c(\omega) \neq 1$, synchronism is impaired [18]; consequently, $m'(n) \nrightarrow m(n)$. Because of the nonlinear nature of the nodes, if any spectral component is amiss, all spectral components at the receiver can be affected.

A way of circumventing these difficulties for ideal but bandlimited channels is to adjust the spectrum of the transmitted chaotic signal to the channel band. A block diagram of the proposed system is shown in Fig. 1. Consider $H_s(\omega)$ as an N-th order finite impulse response filter placed in the path of x_1. Then, for the input $x_1(n)$, its output $x_{k+1}(n)$ is expressed as $x_{k+1}(n) = \sum_{j=1}^{N+1} c_j x_1(n-j+1)$ where c_1, c_2, ..., c_{N+1} are the filter coefficients. Here, the transmitted signal s is obtained by $s(n) = c(x_{k+1}(n), m(n))$. This same filter is placed in the path of y_1. Therefore, this filter output is $y_{k+1}(n) = \sum_{j=1}^{N+1} c_j y_1(n-j+1)$ and $m'(n) = d(y_{k+1}(n), r(n))$. Thus, the dimension of the system of difference equations describing master and slave are now of order $2(k+N)$ instead of $2k$. Provided that the spectrum of $s(n)$ is inside the filter passband and the master obeys the condition (4), then the message can be fully recovered.

As an example, take as chaos generator the two-dimensional Hénon map (e.g. [5], [19])

$$x_1(n+1) = 1 - \alpha x_1^2(n) + x_2(n) \tag{7}$$
$$x_2(n+1) = \beta x_1(n). \tag{8}$$

where α and β are constants. In this case, the master can be written in the form of Eq. (5) with $A_{(N+2)\times(N+2)}$ as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ \beta & 0 & 0 & 0 & \ldots & 0 & 0 \\ c_2 & c_1 & 0 & c_3 & \ldots & c_N & c_{N+1} \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ & & \vdots & & \ddots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{9}$$

The eigenvalues of this matrix are $\lambda_1 = -\lambda_2 = \sqrt{\beta}$ and $\lambda_3 = \ldots = \lambda_{N+2} = 0$. So *chaotic synchronization is guaranteed for $\beta < 1$ independently of the filter coefficients and of $m(n)$.* This is a relevant result: synchronization is not a concern when projecting $H_s(\omega)$. However, as pointed out by [18], a question that must be addressed is whether the generated signals remain chaotic after the introduction of the filter. In the next section, this problem is numerically investigated.

III. Filter Characteristics and Chaos

In principle, there is no guarantee that the system presented in the previous section continues to generate chaotic signals. Consider such a system with linear phase low-pass filters designed by using the window method with discrete-time cut-off frequency ω_c. Assume that $s(n) = c(x_3(n), m(n)) \approx x_3(n)$; that is, the message does not significantly affect the transmitted chaotic signal. This is a reasonable hypothesis, considering practical secure chaos-based communication. The goal is to determine if x_3 remains chaotic after introducing these filters.

The largest Lyapunov exponent h can be used as a measure of chaos (e.g. [7]). Positive h means chaos. This exponent was numerically evaluated by using signals with 5000 points. Figure 2 shows the value of h corresponding to the transmitted signal as a function of the normalized ω_c for filters with $N =$ 10, 20, 50 and 100. The parameter values of the Hénon map were fixed as $\alpha = 0.9$ and $\beta = 0.3$. These choices, supported

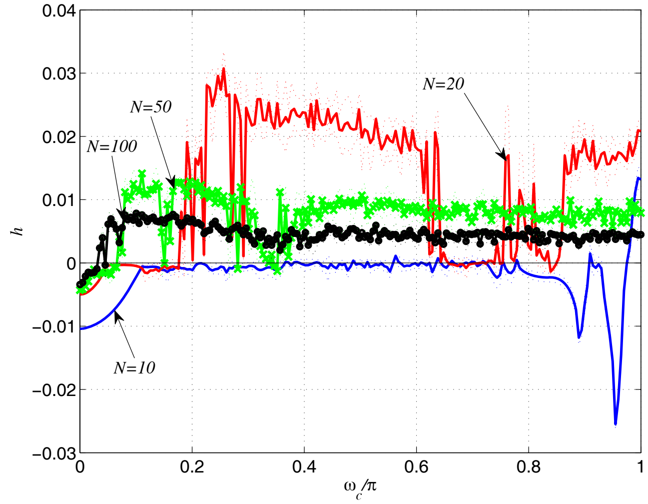

Fig. 2. Largest Lyapunov exponent h in function of the normalized cut-off frequency ω_c of the master filter for different filter orders N. The solid lines represent the average values; the dashed lines, the intervals determined from the standard deviations.

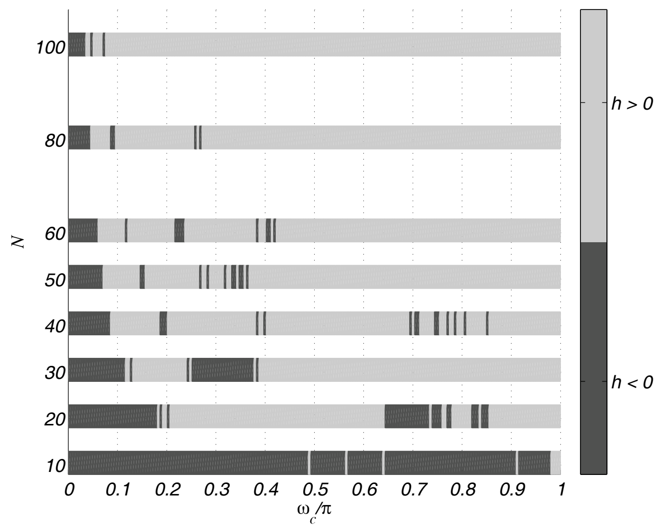

Fig. 3. Chaotic (light gray) and non-chaotic (dark gray) regions for different cut-off frequencies ω_c and filter orders N.

by numerical tests, are convenient for obtaining chaotic signals for several cut-off frequencies and filter orders.

Notice that h is positive in a large range of ω_c for the most of the tested filter orders, but negative values also appear (this implies that $s(n)$ is not chaotic; in fact, it is periodic in such cases). This and other numerical simulations reveal that the higher the filter order, the larger the cut-off frequency range where chaotic behavior can be found. A resume of these results is presented in Fig. 3 ($h > 0$ is represented by light gray and $h < 0$ by dark gray).

These experiments show that it is possible to generate bandlimited chaotic signals by using this synchronization scheme.

IV. Conclusions

The impact of inserting filters in a chaos-based communication scheme operating in bandlimited channel was analytically and numerically investigated. For master and slave governed by the Hénon map, it was analytically shown that the filter coefficients do not influence synchronization. Also, the generated signals are chaotic for a large set of filter orders and cut-off frequencies, according to numerical experiments. These are relevant results if this synchronization scheme is intended to be used for practical communication. The effect of noise on synchronization in our chaos-based scheme remains to be investigated.

V. Acknowledgments

ME and LHAM are partially supported by CNPq; RDF was supported by CAPES.

References

[1] A. Gersho and B. J. Karafin, "Mutual synchronization of geographically separated oscillators," *Bell Syst. Tech. J.*, vol. 45, no. 10, pp. 1689-1704, Dec. 1966.

[2] M. Sexton and A. Reid, *Transmission Networking: SONET and Synchronous Digital Hierarchy*. Artech House, 1992.

[3] W. C. Lindsey, F. Ghazvinian, W. C. Hagmann, and K. Dessouky, "Network synchronization," *Proc. IEEE*, vol. 73, no. 10, pp. 1445-1467, Oct. 1985.

[4] L. H. A. Monteiro, R. V. Santos, and J. R. C. Piqueira, "Estimating the critical number of slave nodes in a single-chain PLL network," *IEEE Commun. Lett.*, vol. 7, no. 9, pp. 449-450, Sep. 2003.

[5] X. Li and G. Chen, "Synchronization and desynchronization of complex dynamical networks: an engineering viewpoint," *IEEE Trans. Circuits Syst. I*, vol. 50, no. 11, pp. 1381-1390, Nov. 2003.

[6] A. Argyris, D. Syvridis, L. Larger, V. Annovazzi-Lodi, P. Colet, I. Fischer, J. Garcia-Ojalvo, C. R. Mirasso, L. Pesquera, and K. A. Shore, "Chaos-based communications at high bit rates using commercial fibre-optic links," *Nature*, vol. 438, no. 7066, pp. 343-346, Nov. 2005.

[7] K. T. Alligood, T. D. Sauer, and J. A. Yorke, *Chaos: An Introduction to Dynamical Systems*. Springer, 1997.

[8] W. M. Tam, F. C. M. Lau, and C. K. Tse, *Digital Communications with Chaos: Multiple Access Techniques and Performance*. Elsevier, 2006.

[9] L. M. Pecora, T. L. Carroll, G. A. Johnson, D. J. Mar, and J. F. Heagy, "Fundamentals of synchronization in chaotic systems, concepts, and applications," *Chaos*, vol. 7, no. 4, pp. 520-543, Dec. 1997.

[10] M. Feki, B. Robert, G. Gelle, and M. Colas, "Secure digital communication using discrete-time chaos synchronization," *Chaos Soliton Fract.*, vol. 18, no. 4, pp. 881-890, Nov. 2003.

[11] K. M. Cuomo and A. V. Oppenheim, "Circuit implementation of synchronized chaos with applications to communications," *Phys. Rev. Lett.*, vol. 71, no. 1, pp. 65-68, July 1993.

[12] C. W. Wu and L. O. Chua, "A simple way to synchronize chaotic systems with applications to secure communication systems," *Int. J. Bifurcat. Chaos*, vol. 3, no. 6, pp. 1619-1627, Dec. 1993.

[13] M. Eisencraft, D. M. Kato, and L. H. A. Monteiro, "Spectral properties of chaotic signals generated by the skew tent map," *Signal Process.*, vol. 90, no. 1, pp. 385-390, Jan. 2010.

[14] Y. Wang, P. Li, and J. Zhang, "Fast random bit generation in optical domain with ultrawide bandwidth chaotic laser," *IEEE Photon. Technol. Lett.*, 2010.

[15] C. Williams, "Chaotic communications over radio channels," *IEEE Trans. Circuits Syst. I*, vol. 48, no. 12, pp. 1394-1404, Dec. 2001.

[16] N. F. Rulkov and L. S. Tsimring, "Synchronization methods for communication with chaos over bandlimited channels," *Int. J. Circuit Theory Appl.*, vol. 27, no. 6, pp. 555-567, Dec. 1999.

[17] M. Eisencraft and M. Gerken, "Comunicação utilizando sinais caóticos: influência de ruído e de limitação em banda," in *Proc. 18th Brazilian Telecommunications Symposium*, vol. 1, pp. 1-6, Sep. 2000.

[18] M. Eisencraft, R. D. Fanganiello, and L. A. Baccala, "Synchronization of discrete-time chaotic systems in bandlimited channels," *Math. Prob. Eng.*, vol. 2009, paper no. 207971, pp. 1-12, 2009.

[19] M. Hénon, "A two-dimensional mapping with a strange attractor," *Commun. Math. Phys.*, vol. 50, no. 1, pp. 69-77, Feb. 1976.

[20] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*. Dekker, 1992.