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Estimation of chaotic signals with applications in communications

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Abstract: In this paper we extend an estimation technique based on Viterbi algorithm for discrete-time chaotic signals immersed in noise. The proposed modification allows for orbits generated by maps with nonuniform invariant density to be estimated. This modified Viterbi algorithm is used in two digital modulation schemes: the Modified Maximum Likelihood Chaos Shift Keying using one and two maps. Both have better symbol error rate characteristics than non-coherent chaos communication schemes.

Keywords: Chaos theory; Communication systems; Estimation theory; Difference equations; Digital communications

1. INTRODUCTION

Recently, various digital modulations using chaotic carriers have been proposed, e.g. [Kennedy et al., 2000b, Lau and Tse, 2003, Stavroulakis, 2005] and references therein. Among them, Chaos Shift Keying (CSK) and its variants based on noncoherent or differential demodulation are promising for practical applications [Kolumbán et al., 1998a,b, Kennedy et al., 2000a,b, Lau and Tse, 2003, Stavroulakis, 2005]. However, these systems still have lower performance than conventional equivalent ones when it comes to symbol error rate in Additive White Gaussian Noise (AWGN) channel [Kennedy et al., 2000b, Lau and Tse, 2003, Xiaofeng et al., 2004, Escribano et al., 2006]. This happens basically because the receiver does not use information on the chaotic maps that generate the transmitted signals. The performance of these systems would be basically the same if one uses random instead of chaotic signals.

An approach for improving these results is to use estimation techniques to counter channel noise before demodulation [Kisel et al., 2001, Xiaofeng et al., 2004, Escribano et al., 2006].

An interesting possibility is the Viterbi Algorithm (VA) as proposed originally by Kisel et al. [2001]. It interprets the chaotic signals as Markov processes that, at each instant, assume one of N_S possible states defined by a partition of the domain U in N_S subintervals. Kisel et al. [2001] used an uniform partition, which is a suitable choice for maps with uniform invariant density. Here we extend these results for general one-dimensional maps using adequate partitions. Our proposal is called the *Modified VA* (MVA).

The objectives of this paper are twofold: i) to describe the MVA and ii) to study the application of MVA in communication systems, like the Maximum-Likelihood CSK (ML-CSK) proposed by Kisel et al. [2001]. ML-CSK can be seen as an identification problem since in the receiver it is necessary to identify the map used to transmit each symbol.

The paper is organized as follows. In Section 2 we describe the MVA. In Section 3 we present the application of estimation methods in identification of chaotic systems and its possible uses in chaotic communications. Finally, Section 4 deals with our conclusions.

2. THE MODIFIED VITERBI ALGORITHM FOR ESTIMATING CHAOTIC SIGNALS

The estimation problem treated here can be stated as follows. An N-point sequence $s'(n)$ is observed. It is modeled as

$$s'(n) = s(n) + w(n),\ 0 \le n \le N-1, \tag{1}$$

where $s(n)$ is an orbit of a one-dimensional system

$$s(n) = f(s(n-1)) \tag{2}$$

and $w(n)$ is AWGN with variance σ^2. The map $f(.)$ is defined in an interval U. We seek an estimate $\hat{s}(n)$ of the orbit $s(n)$.

Consider the domain U as the reunion of disjoint intervals U_j, $j = 1, 2, \ldots, N_S$. In a given instant n, we define that the signal *state* is $q(n) = j$ if $s(n) \in U_j$. A $(k+1)$-length state sequence is represented by

$$\mathbf{q}_k = [q(0), q(1), \ldots, q(k)]^T \tag{3}$$

and the first $k+1$ observed samples by

$$\mathbf{s}'_k = [s'(0), s'(1), \ldots, s'(k)]^T. \tag{4}$$

To simplify notation, we define the N-length sequences $\mathbf{q}_{N-1} \equiv \mathbf{q}$ and $\mathbf{s}'_{N-1} \equiv \mathbf{s}'$. Furthermore, the center of the interval U_j is denoted by $B(j)$.

Given $\mathbf{s}'$, we seek an estimated state sequence $\hat{\mathbf{q}}$ that maximizes the posterior probability

$$P(\hat{\mathbf{q}}|\mathbf{s}') = \max_{\mathbf{q}} P(\mathbf{q}|\mathbf{s}'). \quad (5)$$

Using Bayes' theorem,

$$P(\mathbf{q}|\mathbf{s}') = \frac{p(\mathbf{s}'|\mathbf{q})P(\mathbf{q})}{p(\mathbf{s}')}, \quad (6)$$

where $p(\mathbf{s}')$ and $p(\mathbf{s}'|\mathbf{q})$ are, respectively, the Probability Density Function (PDF) of $\mathbf{s}'$ and the PDF of $\mathbf{s}'$ given that the state sequence of the signal is $\mathbf{q}$. The probability $P(\mathbf{q})$ is the chance of obtaining the state sequence $\mathbf{q}$ when $f(.)$ is iterated.

This way, we must find the argument $\hat{\mathbf{q}}$ so that

$$\hat{\mathbf{q}} = \arg\max_{\mathbf{q}} P(\mathbf{q}|\mathbf{s}') = \arg\max_{\mathbf{q}} p(\mathbf{s}'|\mathbf{q})P(\mathbf{q}). \quad (7)$$

It is important to notice that because of the form as the signals are generated and of the AWGN model, $\mathbf{q}_k$ is a first order Markov process when we consider k as time variable. So,

$$P(\mathbf{q}_k) = P(q(k)|q(k-1))\, P(\mathbf{q}_{k-1}), \quad (8)$$

where $P(q(k)|q(k-1))$ is the transition probability from the state $q(k-1)$ to $q(k)$.

Besides, taking into account the independency of noise samples,

$$p(\mathbf{s}'_k|\mathbf{q}_k) = \prod_{n=0}^{k} p(s'(n)|q(n)) = \prod_{n=0}^{k} p_w(s'(n) - s(n)) \approx \prod_{n=0}^{k} p_w(s'(n) - B(q(n))), \quad (9)$$

where $p_w(.)$ is the PDF of the noise. The approximation in Eq. (9) is valid only if N_S is sufficiently large.

Using Eqs. (7-9), we can express $P(\mathbf{q}|\mathbf{s}')$ as a product of state transition probabilities by conditional observation probabilities. Thus we conclude that $\hat{\mathbf{q}}$ is the sequence that maximizes

$$\left(\prod_{n=1}^{N-1} P(q(n)|q(n-1))\, p(s'(n)|q(n))\right) P(q(0)). \quad (10)$$

Choosing the partition U_j, $j = 1, 2, \ldots, N_S$ so that the probability of each possible state $q(n) = j$ is the same for all j, the last term in Eq. (10), $P(q(0))$, can be eliminated and we get

$$\hat{\mathbf{q}} = \arg\max_{\mathbf{q}} \prod_{n=1}^{N-1} P(q(n)|q(n-1))\, p(s'(n)|q(n)), \quad (11)$$

as Kisel et al. [2001] do. Note, however, the importance of the choice of the partition in obtaining this result. As far as we know, this fact was not noticed before.

To find $\mathbf{q}$ that maximizes the product in Eq. (11) is a classic problem for which an efficient solution is given by VA [Viterbi, 1967], applied to the estimation of chaotic signals for the first time by Marteau and Abarbanel [1991]. Using VA avoids doing an exhaustive search on the $(N_S)^N$ possible state sequences for an N-point signal.

Let $\gamma(n, j)$ be the probability of the most probable state sequence, in the maximum likelihood sense, that ends in state j, at instant $n \geq 1$, given the observed sequence $\mathbf{s}'$, or

$$\gamma(n, j) = \max_{\mathbf{q}_n} P(\mathbf{q}_{n-1}, q(n) = j|\mathbf{s}'). \quad (12)$$

Using Eqs. (8-9), $\gamma(n, j)$ can be calculated in the recursive form

$$\gamma(n, j) = \max_{i} [\gamma(n-1, i)a_{ij}]\, b_j(s'(n)), \quad (13)$$

for $n > 1$ where

$$a_{ij} = P(q(n) = j|q(n-1) = i) \quad (14)$$

and

$$b_j(s'(n)) = p(s'(n)|q(n) = j). \quad (15)$$

The coefficients a_{ij} are the state transition probabilities that depend on the map $f(.)$ and on the partition. We define the transition probabilities matrix as

$$\mathbf{A}_{N_S \times N_S} = a_{ij}, 1 \leq i, j \leq N_S. \quad (16)$$

The coefficients $b_j(.)$ are the observation conditional probabilities that depend only on the PDF of the noise $p_w(.)$.

VA works in two passes, the forward and the backward one:

- **Forward pass**: for each instant $1 \leq n \leq N-1$, Eqs. (12 - 13) are used to calculate $\gamma(n, j)$ for the N_S states. Among the N_S paths that can link states $j = 1, \ldots, N_S$ at instant $n-1$ to state j at instant n, only the most probable one is maintained. The matrix $\varphi(n, j)$, $n = 1, \ldots, N-1$, $j = 1, \ldots, N_S$, stores the state at instant $n-1$ that takes to state j with maximal probability. In the end of this step, at instant $n = N-1$, we select the most probable state as $\hat{q}(N-1)$.
- **Backward pass**: for obtaining the most probable sequence, it is necessary to consider the argument i that maximizes Eq. (13) for each n and j. This is done defining

$$\hat{q}(n) = \varphi(n+1, \hat{q}(n+1)),\ n = N-2, \ldots, 0. \quad (17)$$

Once obtained $\hat{q}(n)$, the estimated orbit is given by the centers of the subintervals related to the most probable state sequence,

$$\hat{s}(n) = B(\hat{q}(n)),\ n = 0, \ldots, N-1. \quad (18)$$

To apply the VA it is necessary to choose a partition so that the probability of an orbit point to be in any state is the same, so that $P(q(0))$ in Expression (10) can be eliminated. This means that if a given map has invariant density $p(s)$ [Lasota and Mackey, 1985], we should take N_S intervals $U_j = [u_j; u_{j+1}]$ so that, for every $j = 1, \ldots, N_S$,

$$\int_{u_j}^{u_{j+1}} p(s)ds = \frac{1}{N_S}. \quad (19)$$

Using the ergodicity of chaotic orbits [Lasota and Mackey, 1985], it is possible to estimate $p(s)$ for a given $f(.)$ and consequently to find the correct partition.

The maps taken as examples by Xiaofeng et al. [2004] and Kisel et al. [2001] have uniform invariant density and the authors have proposed the use of equal length subintervals. However, this choice cannot be generalized for an arbitrary one-dimensional map.

The use of VA with the correct partition is called here *Modified Viterbi Algorithm* (MVA).

As illustrative examples, consider the uniform invariant density tent map defined in $U = [-1, 1]$ as

$$f_T(s) = 1 - 2|s| \tag{20}$$

and the nonuniform invariant density quadratic map

$$f_Q(s) = 1 - 2s^2, \tag{21}$$

defined over the same U [Eisencraft and Baccala, 2008]. It can be shown [Lasota and Mackey, 1985] that, the invariant density of these maps are

$$p_T(s) = 1/2 \tag{22}$$

and

$$p_Q(s) = \frac{1}{\pi\sqrt{1-s^2}}, \tag{23}$$

respectively.

An example of orbit for each of these maps and their respective invariant densities are shown in Figures 1 and 2. The partition satisfying Eq. (19) for each case is also indicated when $N_S = 5$.

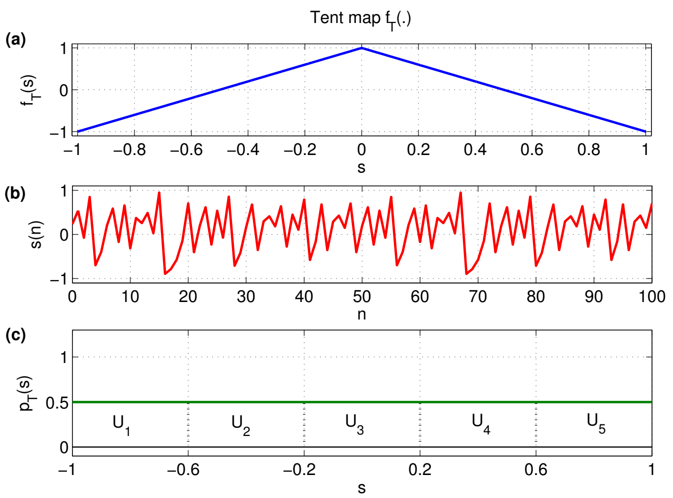

Fig. 1. (a) Tent map $f_T(.)$; (b) example of a 100-point signal generated by $f_T(.)$; (c) invariant density along with the partition satisfying Eq. (19) for $N_S = 5$.

Figures 3 and 4 present how the performance of VA varies for different values of N_S and $N = 10$. In Figure 3 the generating map is $f_T(.)$ whereas $f_Q(.)$ is used in Figure 4. To illustrate the importance of the correct partition choice, Figure 4(a) displays the results of mistakenly using a uniform partition whereas Figure 4(b) displays the results of using the correct partition according to Eq. (19). The input and output SNR are defined as

$$\mathrm{SNR_{in}} = \frac{\sum_{n=0}^{N-1} s^2(n)}{N\sigma^2} \tag{24}$$

and

$$\mathrm{SNR_{out}} = \frac{\sum_{n=0}^{N-1} s^2(n)}{\sum_{n=0}^{N-1} \left(s(n) - \hat{s}(n)\right)^2}. \tag{25}$$

For each $\mathrm{SNR_{in}}$ of the input sequence, the average $\mathrm{SNR_{out}}$ of 1000 estimates is shown.

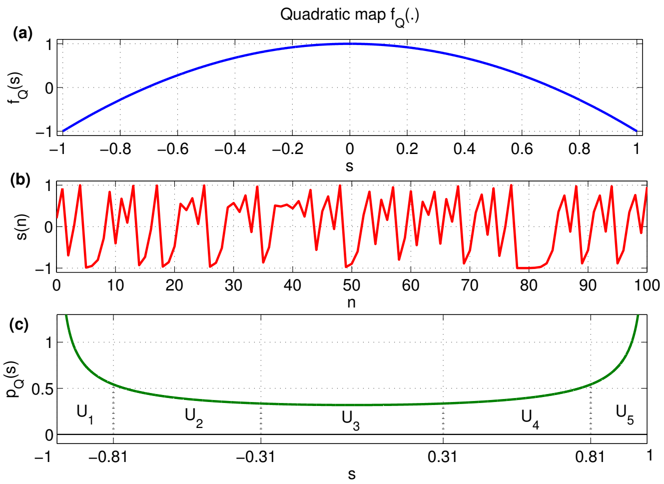

Fig. 2. (a) Quadratic map $f_Q(.)$; (b) example of a 100-point signal generated by $f_Q(.)$; (c) invariant density along with the partition satisfying Eq. (19) for $N_S = 5$.

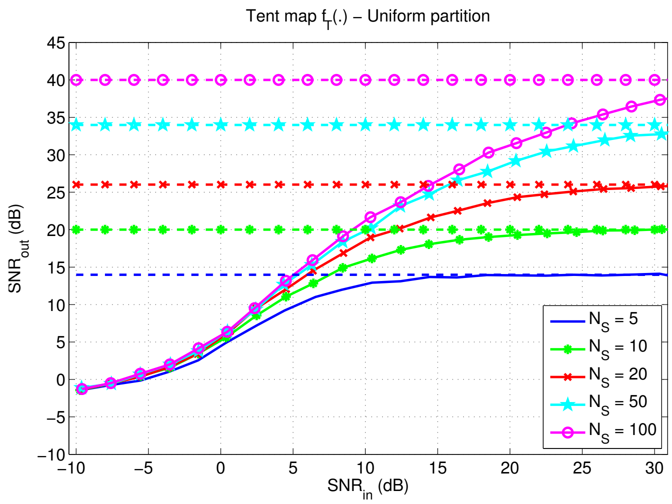

Fig. 3. $\mathrm{SNR_{out}}$ of VA for an orbit of length $N = 10$ using different numbers of partition intervals N_S . The generating map is $f_T(.)$. Performance limits of Eq. (26) are indicated by dashed lines.

Choosing the right partition, the estimation algorithm has an increasing performance as a function of $\mathrm{SNR_{in}}$ until $\mathrm{SNR_{out}}$ attains a limit value which depends on N_S. This limiting value can be calculated assuming that, in the best possible case, the estimation error is caused by domain quantization alone. As such, for an uniform partition, the estimation error is an uniformly distributed random variable in the interval $[-1/N_S, 1/N_S]$. Therefore the mean squared value of $s(n) - \hat{s}(n)$ is limited by $1/\left(3N_S^2\right)$. Additionally, $s(n)$ is uniformly distributed in $[-1, 1]$ and, consequently, has a mean squared value of 1/3. Hence if all the points are in the correct subintervals, the expected value of $\mathrm{SNR_{out}}$, $E[\mathrm{SNR_{out}}]$ in dB is

$$E\left[\mathrm{SNR_{out}}\right] = E\left[10\log\frac{\sum_{n=0}^{N-1} s^2(n)}{\sum_{n=0}^{N-1}(s(n)-\hat{s}(n))^2}\right] = \\ = 10\log\frac{N/3}{N/(3N_S^2)} = 20\log N_S. \tag{26}$$

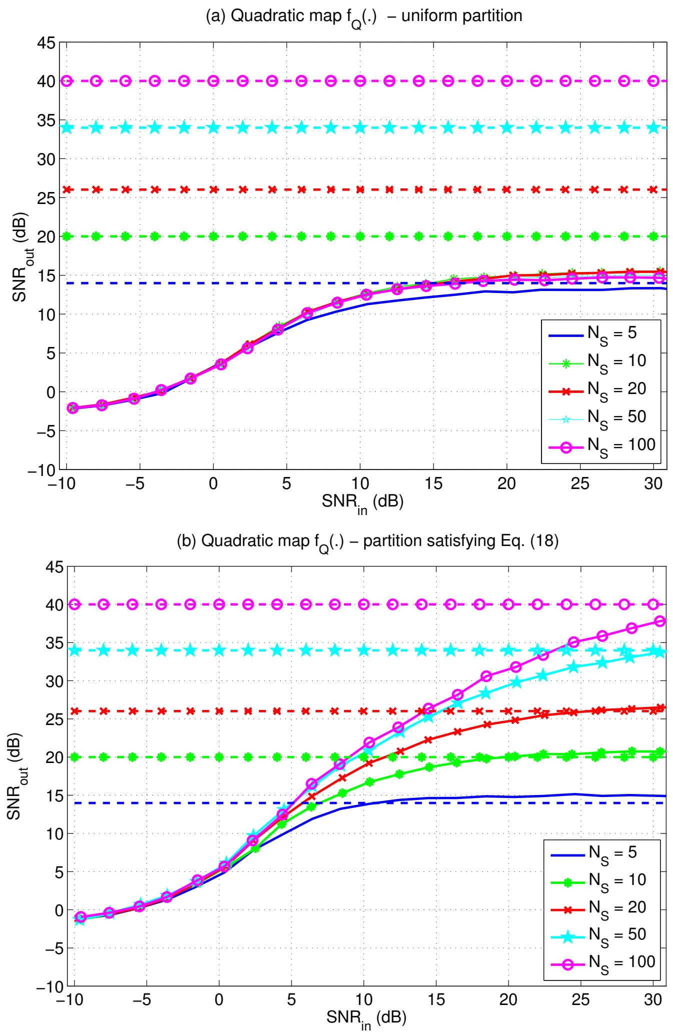

Fig. 4. SNR_{out} of the Viterbi algorithm for an orbit of length $N = 10$ using different number of partition intervals N_S. The generating map is $f_Q(.)$. Results for an uniform partition (a) are contrasted to the improved values in (b) using a partition satisfying Eq. (19). Limits of Eq. (26) are indicated by dashed lines.

These limits, which are exact only in the uniform partition case, are indicated with dashed lines for each N_S value in Figures 3 and 4.

Comparing Figures 4(a) and (b) reveals the critical role played by the partition choice. Clearly the uniform partition of Xiaofeng et al. [2004] and Kisel et al. [2001] cannot attain the best possible SNR_{out} for the quadratic map whose invariant density is not uniform.

Figures 3 and 4(b) show that the algorithm has a slightly better performance for the quadratic map. This result confirms the importance of the choose of map to be used. The modification on VA proposed here permits testing more maps looking for an optimal one.

3. CHAOTIC SYSTEM ESTIMATION AND IDENTIFICATION: APPLICATIONS IN COMMUNICATION

In this section we propose two binary digital modulation using chaotic system identification. They are the *Modified Maximum Likelihood Chaos Shift Keying* (MMLCSK) using one and two maps. Both are based on the ones proposed by Kisel et al. [2001]. We have modified them using nonuniform partitions for the MVA as discussed in the previous section. This way, it is possible to test the performance of nonuniform invariant density maps.

3.1 MMLCSK using two maps

In this case, each symbol is associated with a different map $f_1(.)$ and $f_2(.)$. To transmit a **0**, the transmitter sends an N-point orbit $s_1(.)$ of $f_1(.)$ and to transmit a **1**, it sends an N-point orbit $s_2(.)$ of $f_2(.)$.

Maps must be chosen so that their state transition probabilities matrix $\mathbf{A}_1$ and $\mathbf{A}_2$ are different. Estimating $s_1(n)$ using MVA with $\mathbf{A}_2$ must produce a small estimation gain or even a negative (in dB) one. The same must happen when we try to estimate $s_2(n)$ using $\mathbf{A}_1$.

The receiver for MMLCSK using two maps is shown in Figure 5. The Viterbi decoders try to estimate the original $s(n)$ using $\mathbf{A}_1$ or $\mathbf{A}_2$. For each symbol, the estimated state sequences are $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$.

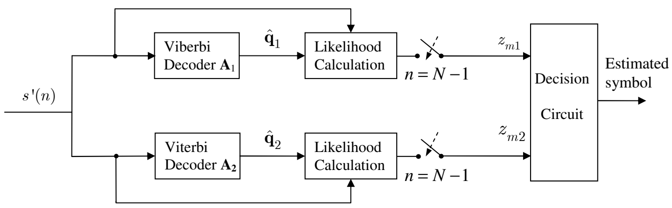

Fig. 5. Receiver for MMLCSK using two maps.

Given the observed samples, z_{m1} e z_{m2} are proportional to the probability of obtaining $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ respectively. More precisely,

$$z_{m1} = \prod_{n=1}^{N-1} P(\hat{q}_1(n)|\hat{q}_1(n-1), \mathbf{A}_1) p(s'(n)|\hat{q}_1(n)), \quad (27)$$

$$z_{m2} = \prod_{n=1}^{N-1} P(\hat{q}_2(n)|\hat{q}_2(n-1), \mathbf{A}_2) p(s'(n)|\hat{q}_2(n)). \quad (28)$$

In this equations we have used the likelihood measure of Eq. (11). The probability $P(\hat{q}(n)|\hat{q}(n-1), \mathbf{A}_i)$ can be read directly from $\mathbf{A}_i$ and $p(x'(n)|\hat{q}(n))$ depends only on the noise and can be approximated as described by Dedieu and Kisel [1999].

Choosing the largest between z_{m1} e z_{m2} we can *identify* the map used in the transmitter with maximum likelihood and, this way, decode the transmitted symbol.

Given a map $f_1(.)$, an important problem is to find a map $f_2(.)$ so that its probability transition matrix $\mathbf{A}_2$ permits to discriminate between the likelihood measures of Eqs. (27) and (28). For piecewise linear maps on the interval $U = [-1, 1]$ we can use the following rule adapted from [Kisel et al., 2001]:

$$f_2(s) = \begin{cases} f_1(s) + 1, & f_1(s) < 0 \\ f_1(s) - 1, & f_1(s) \geq 0 \end{cases}. \quad (29)$$

Figure 6 shows the construction of map $f_2(.)$ from $f_1(.) = f_T(.)$. This way, $f_1(s)$ and $f_2(s)$ map a point s a unity away.

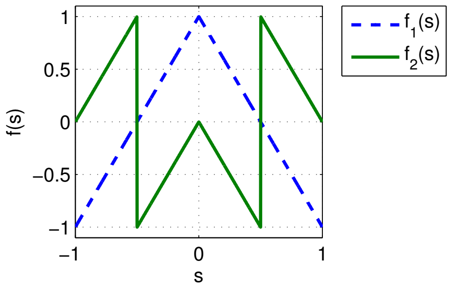

Fig. 6. Construction of map $f_2(.)$ for $f_1(.) = f_T(.)$ using Eq. (29).

In this case, using an uniform partition for $N_S = 5$ we have

$$\mathbf{A}_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}. \tag{30}$$

It can be shown that almost every orbit generated by $f_2(.)$ are in fact chaotic [Kisel et al., 2001]. Note however that this method is not necessarily optimal and must be used prudently. There is no guaranty that the orbits of $f_2(.)$ given by Eq. (29) are chaotic in general.

For instance, if we apply the same strategy for the quadratic map $f_1(s) = f_Q(s)$, we obtain $f_2(s)$ show in Figure 7. All the orbits of $f_2(.)$ converge to a stable fixed point at $s = 0$ and hence are not chaotic at all [Devaney, 2003].

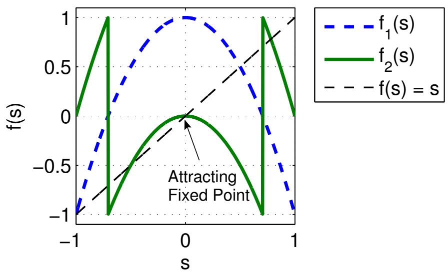

Fig. 7. Construction of $f_2(.)$ for $f_1(.) = f_Q(.)$ using Eq. (29). Note the attracting fixed point.

In the simulations, we have used $f_2(.) = -f_Q(.)$ shown in Figure 8. This map is possibly not optimum because points next to the roots of $f_1(.)$ and $f_2(.)$ are mapped near to each other by both functions. The transition matrix for these two maps for $N_S = 5$ using the partition obeying Eq. (19) are

$$\mathbf{A}_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}. \tag{31}$$

In this case, it can be shown that $f_2(.)$ generates chaotic orbits [Devaney, 2003]. However, note that a_{23} e a_{43} exhibit nonzero probabilities in both matrices what will probably generate errors in the MMLCSK receptor. This

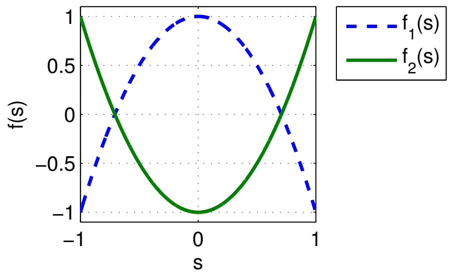

Fig. 8. Construction of $f_2(.)$ for $f_1(.) = f_Q(.)$ used in simulations.

way, we expect a worst performance of this couple of maps when compared to the one with matrices given by Eq. (30).

To find $f_2(.)$ given a map $f_1(.)$ that presents optimal properties when it comes to identification throw the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ is an open problem. As shown by the last example, it is necessary to impose that $f_2(.)$ generates chaotic orbits.

3.2 MMLCK using one map

As an alternative, it is possible to construct a communication system based on MVA estimation using just one map. In this case, according to the symbol that is intended to be communicated, the chaotic signal is directly transmitted or an invertible transformation is applied on the sequence. This operation must modify the sequence so that it is no more a valid orbit of the used map. This way, it is no longer necessary to find a map $f_2(.)$.

In the binary case, for maps that are not odd, this transformation can be, for instance, $T(s) = -s$ witch can be undone multiplying again the sequence by -1. To transmit a **0**, we send an N-point orbit $s_1(.)$ of $f_1(.)$. To transmit a **1**, we send $-s_1(.)$.

The receiver for this system is shown in Figure 9. The variables z_{m1} and z_{m2} are calculated by Eq. (27). However, when calculating z_{m2}, $s'(n)$ is substituted by $-s'(n)$. So, when a **0** is received, the likelihood expressed by z_{m1} must be greater than z_{m2} because $-s_1(n)$ is not an orbit of $f_1(.)$. The opposite is true when a **1** is received.

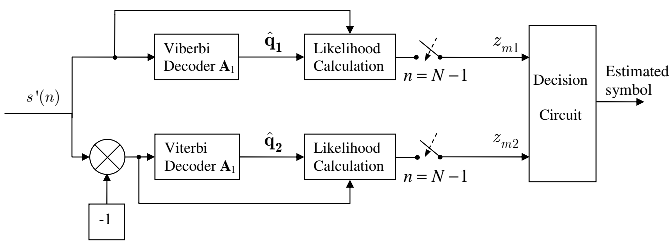

Fig. 9. Receiver for MMLCSK using two maps.

It is relevant to note that this scheme can be easily generalized for an M-ary modulation, $M > 2$. For this it is necessary just to consider other invertible transformations.

3.3 Numerical simulations

Figure 10 show the Symbol Error Rate (SER) as a function of the Bit Energy per Power Spectral Density of the

AWGN (E_b/N_0) for the MMLCSK using one and two maps. In the estimation and identification process it was considered $N_S = 100$ subsets and $N = 50$ samples per bit. For sake of comparison, it is also shown the performance of Chaos On-Off Keying (COOK) [Kolumbán et al., 1998a], the non-coherent chaos communication that does not use estimation with best performance. This system is based only on energy estimation to decode the signal.

Our simulations show that MMLCSK using one map has a slightly better performance than MMLCSK using two maps. Besides $f_T(.)$ performs better than $f_Q(.)$. This last results confirms the importance of the choose of map and transform to be employed.

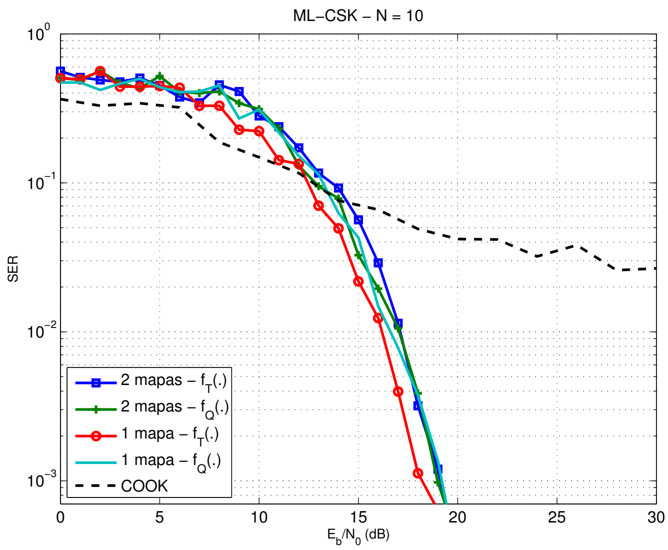

Fig. 10. Symbol Error Rate (SER) for the tested MMLCSK modulations. Each bit was represented by $N = 10$ samples.

4. CONCLUSIONS

In this paper we present a generalization of Viterbi algorithm proposed by Kisel et al. [2001], the Modified Viterbi Algorithm (MVA). The modification consists of using nonuniform partition of the domain to allow for orbits of maps with nonuniform invariant density to be correctly estimated.

Following, MVA was applied in the identification of chaotic maps from its orbits and its application in communications. It was proposed the MMLCSK, which is the MLCSK using MVA. Compared with methods that do not use estimation, these systems performs better in terms of SER in AWGN. The cost of this improvement is the complexity of the receiver. However, the information is coded on the dynamic of the chaotic system and not in easily measured properties like energy, what makes it more difficult to be detected without authorization. It is necessary to know the state transitions matrices to demodulate.

More research is necessary in order to find optimal couple maps and transforms to optimize the discrimination between maps in the receiver.

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