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Spectral properties of chaotic signals with applications in communications

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ABSTRACT

This paper investigates the characteristics of the Power Spectral Density (PSD) of chaotic signals generated by skew tent maps. The influence of the Lyapunov exponent on the autocorrelation sequence and on the PSD is evaluated via computational simulations. We conclude that the essential bandwidth of these signals is strongly related to this exponent and they can be low-pass or high-pass depending on the family's parameter. This way, the PSD of a chaotic signal is a function of the generating map although this is not a one-to-one relationship.

1. Introduction

A chaotic signal is defined as being deterministic, aperiodic and presenting sensitivity to initial conditions. This last property means that, if the generator system is initialized with a slightly different initial condition, the obtained signal quickly diverges from the original one [1].

From the Telecommunication Engineering point of view, chaotic signals possess some interesting properties. The literature, e.g. [2,3], uses to consider that they have broadband, impulsive Autocorrelation Sequence (ACS) and that the cross-correlation sequence between signals with different initial conditions assumes low values. Due to these characteristics, since the beginning of the 1990s, the field of communication with chaotic carriers has received a great deal of attention, see e.g. [2,4] and the references therein.

Using chaotic signals to modulate narrowband information signals results in larger bandwidth and lower Power Spectral Density (PSD) level, which characterize spread spectrum systems [5]. This way, chaotic modulations possess the same qualities than conventional spread spectrum [2], mitigating both multipath and jamming effects.

The study of spectral characteristics is an important issue when it comes to using chaotic signals in practical communications. A great deal of the existing technology is based on frequency multiplexing and, besides, the bandwidth of the transmitted signal is an essential parameter when planning a communication system. Some works such as [6–12] depict the PSD of continuous-time chaotic signals generated by particular systems. Ullmann and Caldas [13] have used the PSD of discrete-time chaotic signals to study transitions in the parameter space. Callegari et al. have studied the prediction of some spectral properties of frequency modulated chaotic signals [14]. However, the spectral characteristics of discrete-time chaotic signals have been rarely deeply studied. Most papers just state that they are broadband signals and sometimes use this property to characterize them, e.g. [15].

Although this broadband characteristic is present in commonly used maps, as the tent [16] and the logistic ones [1], it is not a necessary condition for chaos. In this work we show that it is possible to easily generate low-pass or high-pass chaotic signals.

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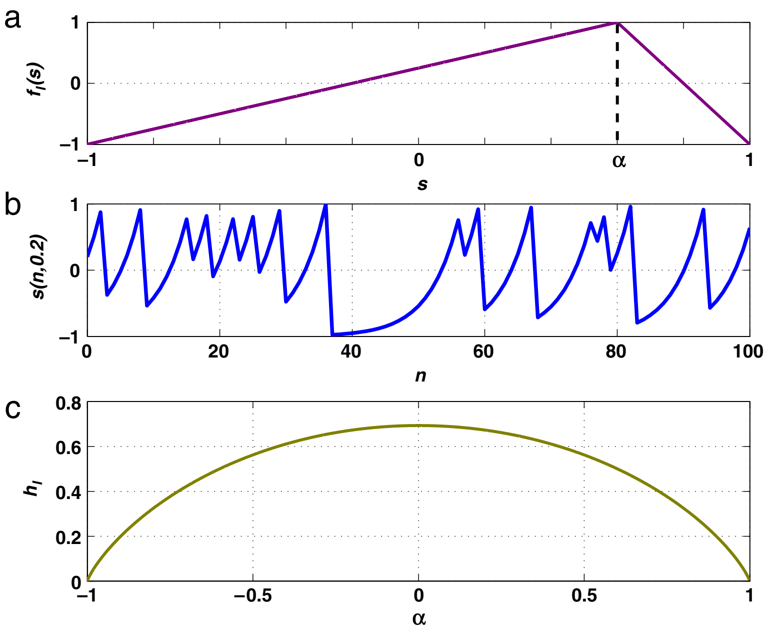

Fig. 1. (a) Skew tent map $f_l(s)$; (b) the orbit $s(n, 0.2)$ for $\alpha = 0.6$ and (c) Lyapunov exponent of the chaotic orbits as a function of α.

The objective of this paper is to present some preliminary results [17] on this far-reaching subject and to probe some possible applications. We investigate via computational simulation the PSD of discrete-time chaotic signals generated by a family of piecewise linear maps, the skew tent one. Furthermore, we relate a property of the chaotic attractor of these orbits, the *Lyapunov exponent* [1] with a convenient measure of the bandwidth, the *essential bandwidth* [5].

The paper is organized as follows. Section 2 presents the skew tent family and its relevant characteristics. Techniques for obtaining the PSD of chaotic signals are discussed in Section 3. In Section 4 the relationship between Lyapunov exponent and essential bandwidth is explored. The results are used to separate chaotic signals in Section 5. Finally, we present some conclusions in Section 6.

2. Skew tent maps

A one-dimensional *discrete-time dynamical system* or *map* is defined by the difference equation

$$s(n+1) = f(s(n)), \tag{1}$$

where $f(.)$ is a function with the same domain and range space $U \subset \mathbb{R}$, $n \in \mathbb{N}$ and $s(0) \in U$. For each *initial condition* s_0, an *orbit* or *signal* becomes defined as $s(n, s_0) = f^n(s_0)$ with $f^n(.)$ being the nth successive application of $f(.)$. For simplicity of notation, an orbit will be symbolized by $s(n)$ whenever s_0 is immaterial.

In this paper, we focus on the skew tent maps defined by

$$s(n+1) = f_l(s(n)) \tag{2}$$

where

$$f_l(s) = \begin{cases} \dfrac{2}{\alpha+1}s + \dfrac{1-\alpha}{\alpha+1}, & -1 < s < \alpha \\[2ex] \dfrac{2}{\alpha-1}s - \dfrac{\alpha+1}{\alpha-1}, & \alpha \le s < 1 \end{cases} \tag{3}$$

and $\{\alpha, s(0)\} \subset U = (-1, 1)$. This family is a modified version of the one proposed in [18]. The parameter α determines the x-coordinate of the tent's peak. This map is shown in Fig. 1(a) along with the orbit $s(n, 0.2)$ for $\alpha = 0.6$ in Fig. 1(b).

The Lyapunov exponent h is the divergence rate between nearby orbits and is usually taken as a measure of the "chaoticness" of an aperiodic signal. For the orbit $s(n, s_0)$, it is given by [1]

$$h = \lim_{N\to\infty} \frac{1}{N} \left(\sum_{n=0}^{N-1} \ln \left| f'\left(s\left(n, s_0\right)\right) \right| \right), \tag{4}$$

where $f'(s)$ is the derivative of $f(s)$ supposed to be defined for all s in its domain. A positive h is a sufficient condition for an aperiodic signal to be classified as chaotic [1].

It can be shown [18] that the Lyapunov exponent of almost every orbit of a skew tent map is a function of α only and is given by

$$h_l = \frac{\alpha+1}{2}\ln\left(\frac{2}{\alpha+1}\right) + \frac{1-\alpha}{2}\ln\left(\frac{2}{1-\alpha}\right). \tag{5}$$

Fig. 1(c) shows how h_l varies with α. For every considered value of α, $h_l > 0$ and the maximum value of h_l, $h_{lmax} = \ln 2$, is attained for $\alpha = 0$.

The chaotic orbits generated by Eq. (2) have uniform invariant density over $(-1, 1)$ [19]. This means that the orbit points are uniformly distributed on this interval. Consequently, these orbits are all zero-mean and their average power is 1/3 independently of α.

In the next two sections we characterize the ACS and the PSD of the signals generated by these maps.

3. PSD of chaotic signals

There are two different ways to interpret chaotic signals generated by a given map [19]. They can be seen as deterministic individual signals or as sample functions of an ergodic stochastic process. Each of these interpretations gives rise to different forms of calculating the PSD. Both are analyzed in this section.

3.1. Chaotic signals as deterministic individual signals

Given the map $f(.)$ in Eq. (1) and the initial condition $s(0) = s_0$, the sequence $s(n, s_0)$ is well defined for all $n \geq 0$ and its ACS can be readily determined as

$$R(l, s_0) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} s(n, s_0)s(n+l, s_0), \tag{6}$$

where l is an integer [20]. In this calculation, we consider $s(n+l, s_0) = 0$ whenever $n+l$ results in a negative number.

The PSD $S(f, s_0)$ is the Discrete-Time Fourier Transform (DTFT) of $R(l, s_0)$, considering l as time variable [20]:

$$S(f, s_0) = \sum_{l=-\infty}^{\infty} R(l, s_0)\,\mathrm{e}^{-j\pi f l}, \tag{7}$$

where f is the normalized frequency: $f = 1$ is equivalent to the discrete frequency $\omega = \pi$.

Fig. 2 shows six representative orbits and their estimated PSD obtained via Eqs. (6)–(7) with $N = 20\,000$. The PSD curves were normalized so that their maximum value is 1.

Based on our computational simulations we can state that:

(i) when the parameter α is positive, the generated signals vary slowly in time and are low-pass, as can be seen in Fig. 2(a) and (b);
(ii) when $|\alpha|$ is next to zero, the generated signals have broadband, as in Fig. 2(c) and (d);
(iii) for negative values of α, the orbits oscillate quickly in time and are high-pass, as in Fig. 2(e) and (f);
(iv) orbits generated by the same map with different initial conditions present similar PSD despite the fact that they are pointwise different in time. This is exemplified by the comparisons of Fig. 2(a) and (b), (c) and (d) or (e) and (f).

This way, the map, defined by α, is determinant in the spectral characteristics of the signals it generates. The spectral similarities between orbits generated by the same map motivates the interpretation of a chaotic signal as a realization of an ergodic stochastic process.

It is relevant to note that for $\alpha = 0$ just irrational initial conditions generate chaotic signals [1]. This way they cannot be obtained directly via iterations of Eq. (2) in this case. The orbits shown in Fig. 2(c) and (d) were plotted exploring the conjugacy [1] between the map $f_l(s)$ with $\alpha = 0$ and the quadratic map

$$s(n+1) = f_Q(s(n)) = -2s^2(n) + 1. \tag{8}$$

3.2. Chaotic signals as sample functions of a stochastic process

Chaotic signals generated by a fixed map can be modeled as an ergodic stochastic process in which each initial condition defines a sample function [19]. This interpretation has the advantage of highlighting properties that apply to the entire set of chaotic orbits defined by the map.

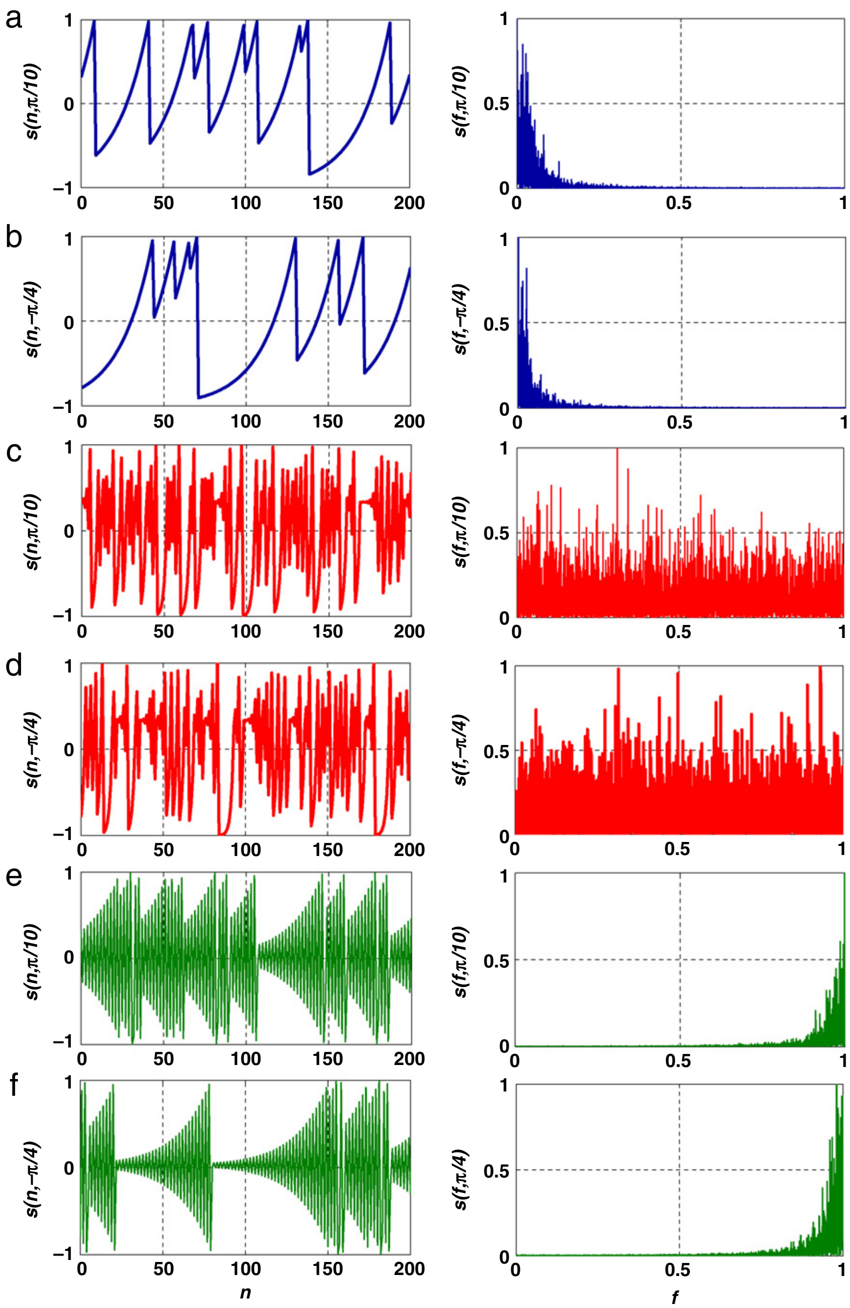

Fig. 2. PSD of individual orbits $s(n, s_0)$: (a) $s_0 = \pi/10$, $\alpha = 0.9$; (b) $s_0 = -\pi/4$, $\alpha = 0.9$; (c) $s_0 = \pi/10$, $\alpha = 0$; (d) $s_0 = -\pi/4$, $\alpha = 0$; (e) $s_0 = \pi/10$, $\alpha = -0.9$; (f) $s_0 = -\pi/4$, $\alpha = -0.9$.

In this case we define the ACS as

$$R_S(l) = E\left[R\left(l, s_0\right)\right], \tag{9}$$

where the expectation $E[.]$ is taken over all initial conditions that generate chaotic orbits. The PSD $S_S(f)$ is the DTFT of $R_S(l)$, as is done with conventional stochastic processes [21].

Fig. 3 shows estimates of the PSD and of the normalized ACS for different values of α. For each curve, the expectation in (9) is estimated considering 20 000 orbits with $N = 440$ samples and initial conditions s_0 uniformly distributed in U.

These simulations show that:

(i) the higher the absolute value of α, the narrower the bandwidth of the generated chaotic signals. This way, changing α, it is possible to obtain low-pass or high-pass chaotic signals with arbitrary bandwidth;

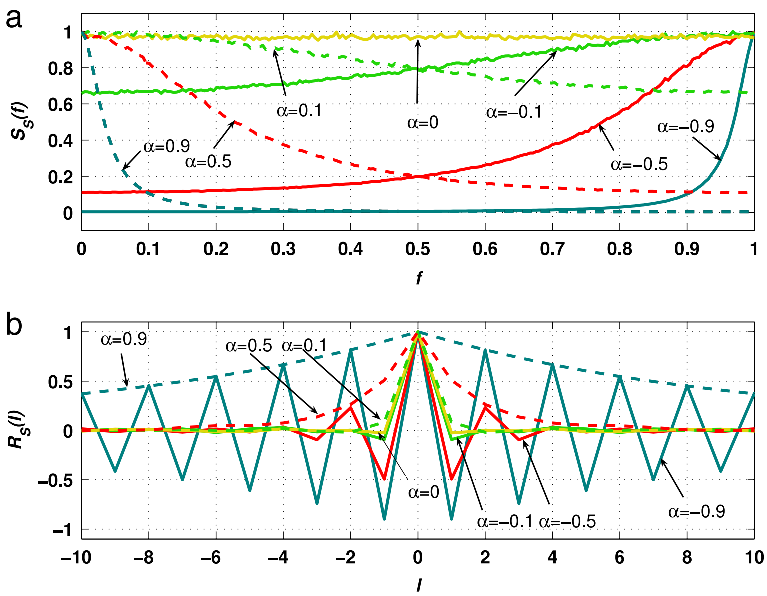

Fig. 3. (a) PSD and (b) ACS of the ensemble of orbits defined by different values of the parameter α. The ACS is normalized so that $R_S(0) = 1$.

(ii) the signal of α defines if the obtained signals are low-pass or high-pass;
(iii) the PSDs of the signals generated by α and $-\alpha$ present symmetry around $f = 0.5$, as can be seen in Fig. 3(a);
(iv) for $\alpha > 0$, $R_S(l)$ is monotonically decreasing with $|l|$. For $\alpha < 0$, $R_S(l)$ oscillates indicating that in this case for almost any n and s_0, the signals of $s(n, s_0)$ and $s(n+1, s_0)$ are different;

It is worth to note that for $\alpha = 0$, the map $f_l(.)$ coincides with the one used in [16] for $\beta = 2$. In this situation, that paper has demonstrated that the generated signals have white spectrum. Our results agree perfectly with theirs.

These results mean that chaos is far away from being a synonym for broadband non-correlated signals. Their spectral characteristics are a function of the generating map. However, this relation is not one-to-one. There are different maps that generate sets of orbits with same PSD. For instance, computational simulations show that $f_Q(.)$ of Eq. (8) generates white spectrum orbits as $f_l(.)$ with $\alpha = 0$ does.

However, when the invariant density of the orbits [19] is considered along with PSD, we conjecture a more restrictive, possible one-to-one, relationship between map and these functions. This is equivalent to the fact that the probability density function and the PSD defines an ergodic conventional stochastic process in a unique way.

4. Essential bandwidth and the Lyapunov exponent

The bandlimiting properties of signals can be measured by the *essential bandwidth* defined as the frequency range where 95% of the total signal power is concentrated [5]. We use here a normalized version of this definition, $0 \leq B \leq 1$, dividing the essential bandwidth by 0.95. Using this definition, a white noise has $B = 1$.

From the curves in Fig. 3(a), we see that the value of B is determined by $|\alpha|$. This control is justified by the direct relationship between this parameter and the Lyapunov exponent shown in Fig. 1(c). The lower the absolute value of α, the higher the value of h_l, which means that the orbits diverge faster from nearby ones and the ACS tends to an impulsive format. Consequently, the PSD, being the DTFT of the ACS, will have a larger B. Fig. 4 presents curves of B as function of $|\alpha|$ and of h_l. One can see that we go from a white spectrum at $\alpha = 0$ ($h_l = \ln 2$) to an extremely narrowband signal at $|\alpha|$ next to the unity (h_l next to zero).

The existence of a one-to-one relationship between B and $|\alpha|$ is significative. Choosing a convenient α, it is possible to generate a low-pass or high-pass chaotic signal with arbitrary essential bandwidth. Besides, it is possible to use the PSD of an observed orbit to estimate α.

5. Example of application: Chaotic signals separation

As a simple application example of the above concepts, we consider the problem of chaotic signal separation using a filter bank. The possibility of generating chaotic signals with well-established essential bandwidth permits the multiplexing of digital modulations using chaotic carriers as the Maximum Likelihood Chaos Shift Keying proposed in [18].

Let the signals $s_1(n)$ and $s_2(n)$ be generated by Eq. (2) with $\alpha = 0.95$ and $\alpha = -0.95$. Possible examples are shown in Figs. 5(a) and 6(a), respectively. This way, $s_1(n)$ is a low-pass signal and $s_2(n)$ is a high-pass signal with essential bandwidth $B \approx 0.21$ (Fig. 4(a)).

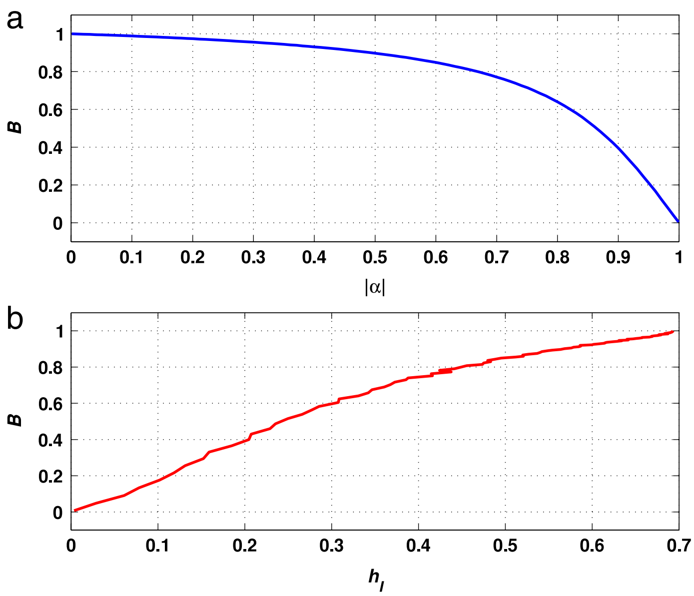

Fig. 4. Essential bandwidth B as a function of (a) $|\alpha|$ and of (b) the Lyapunov exponent h_l.

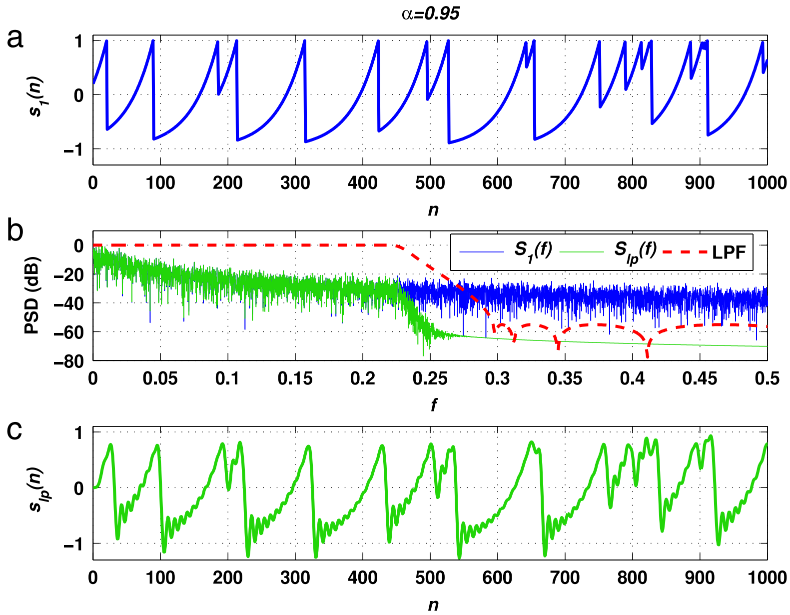

Fig. 5. (a) Signal $s_1(n)$ generated with $\alpha = 0.95$; (b) PSDs of $s_1(n)$ and of $s_{lp}(n)$, its low-pass filtered version. LPF is an 11th-order elliptical low-pass filter; (c) $s_{lp}(n)$.

In spite of the fact that around 95% of the power of these signals are concentrated inside the essential bandwidth, they can have spurious components outside it which would compromise the performance of the separation process using frequency-selective filters.

For these signals to be effectively bandlimited, we can process them using low-pass or high-pass filters with cutoff frequency B. This way, the filtering does not change significantly the signal time and spectrum characteristics. The application of these filters to $s_1(n)$ and $s_2(n)$ giving rise respectively to $s_{lp}(n)$ and $s_{hp}(n)$ is shown in Figs. 5(b) and 6(b) in the frequency domain. The resulting signals are shown in Figs. 5(c) and 6(c).

These bandlimited chaotic signals can be easily frequency multiplexed as is done in conventional communication systems [21]. Baseband signals with non-overlapping spectrum as $s_{lp}(n)$ and $s_{hp}(n)$ can be separated using the same kind of filters used to generate them.

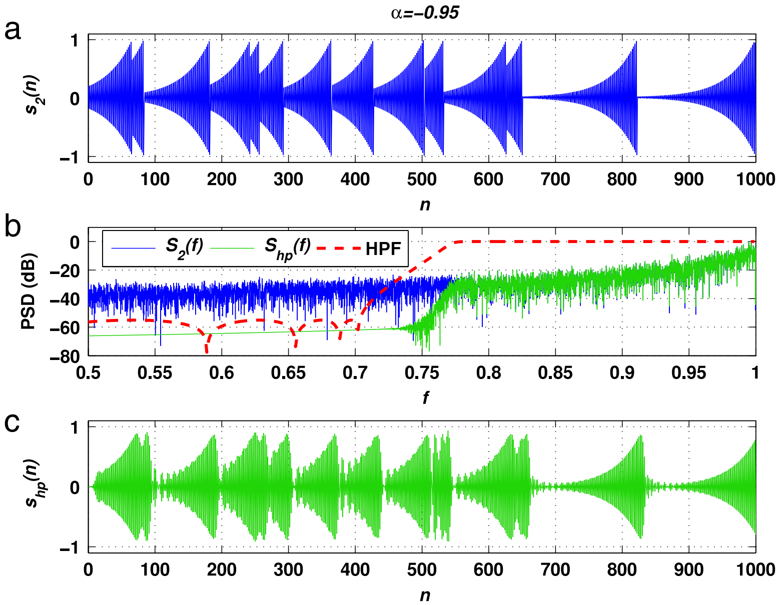

Fig. 6. (a) Signal $s_2(n)$ generated with $\alpha = -0.95$; (b) PSD of $s_2(n)$ and of $s_{hp}(n)$, its high-pass filtered version. HPF is an 11th-order elliptical high-pass filter; (c) $s_{hp}(n)$.

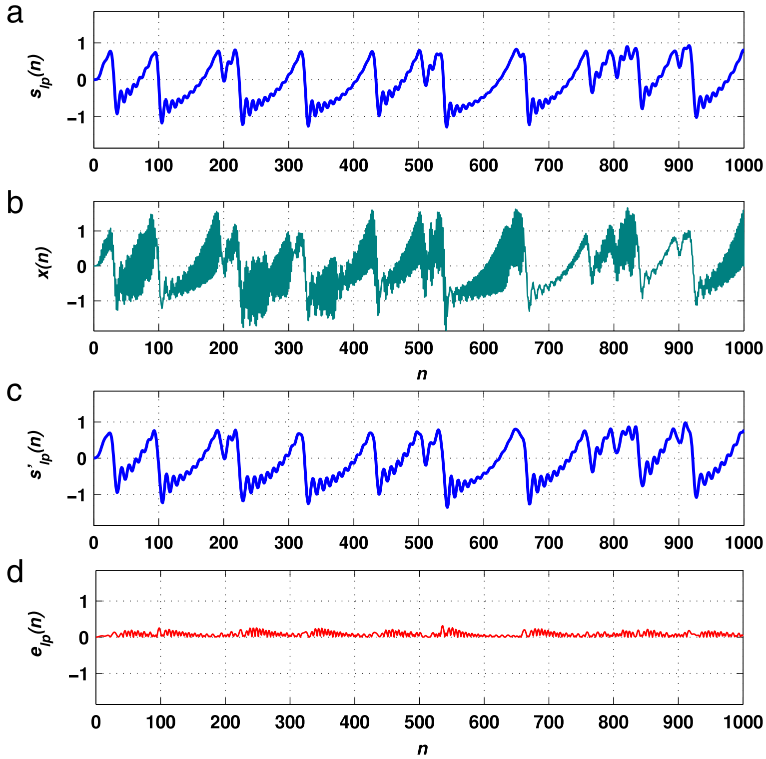

Fig. 7. (a) Signal $s_{lp}(n)$ generated with $\alpha = 0.95$; (b) $x(n) = s_{lp}(n) + s_{hp}(n)$; (c) separated low-pass signal $s'_{lp}(n)$; (d) error signal $e_{lp}(n) = |s_{lp}(n) - s'_{lp}(n)|$.

In Fig. 7(c) we show the resulting separated signal $s'_{lp}(n)$ obtained from

$$x(n) = s_{lp}(n) + s_{hp}(n). \tag{10}$$

The same low-pass filter used to generate $s_{lp}(n)$ is employed in this processing. The signals $x(n)$, $s_{lp}(n)$ and the error

$$e_{lp}(n) = \left| s_{lp}(n) - s'_{lp}(n) \right| \tag{11}$$

are also shown in Fig. 7.

6. Conclusions

This paper analyzes through computational simulations the PSD of the orbits generated by the skew tent maps given by Eq. (2). We have shown the influence of the parameter α and of the Lyapunov exponent on the ACS and PSD of the obtained chaotic signals. Also, a possible application in multiplexing chaotic signals is probed.

An important preliminary consequence of our results is that there are situations when the chaotic signals generated by one-dimensional maps are not broadband. Furthermore, the ACS of these signals is not necessarily impulsive. Fig. 4 suggests that from a specific spectral characteristic, the essential bandwidth, it is possible to find a piecewise linear generator map that produces orbits with the desired B very easily.

The strong relationship between Lyapunov exponent and bandwidth can be used to implement new ideas for receivers in chaotic digital modulation systems. An alternative would be to associate different symbols with different values of $|\alpha|$ and to transmit N points of an orbit generated by the corresponding map. In the receiver the essential bandwidth of the received signal would be estimated and $|\alpha|$ together with the associated symbol would be determined. Following this path, there are many possibilities to explore in future researches.

The generalization of our results to other one-dimensional maps seems to be possible using the *conjugacy* concept [1]. Numerical simulations show that conjugated maps generate orbits with similar spectral characteristics. This subject is under research.

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