

ESTIMATING THE INITIAL CONDITION OF CHAOTIC ORBITS: PERFORMANCE BOUNDS

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Abstract: We derive the Cramér-Rao lower bound associated with the estimation of the initial condition of noise-embedded chaotic signals produced by general one-dimensional maps. We relate its asymptotic behaviour to the chaotic attractor's Lyapunov number. These results can be used to choose the chaotic generator more suitable for applications on chaotic digital communication systems.

Keywords: Chaos theory, Estimation theory, Communication systems, Nonlinear systems, Parameter estimation

1. INTRODUCTION

Over the last decade many results associated with the application of chaotic signals in digital communication have appeared, e.g. (Kennedy et al., 2000; Lau and Tse, 2003) and references therein. In these systems, digital information is mapped directly onto wideband chaotic rather periodic waveforms that play the role of signal carrier. Because of their wideband features, these systems effectively spread spectral energy and thereby potentially mitigate both multipath and jamming effects (Haykin, 2000).

Chaos generating systems are plentiful and it is natural to ask whether one given chaotic signal generator may provide superior performance over others. Whereas this question remains unanswered in general (Lau and Tse, 2003), the restricted, albeit important class of systems using piece-wise linear maps is the most used to generate chaos because of its simplicity. In its use, however,

there is very little concern about the existence of an optimum chaos generating map that can improve performance.

To compare such generic one-dimensional chaos generating maps, we examine the Cramér-Rao lower bound (CRLB) for the estimation of initial conditions of chaotic orbits in the presence of additive observation noise. We relate it to the system's Lyapunov number L often used as its measure of "chaoticness" and whose numerical determination is possible via a variety of techniques (Abarbanel, 1996).

This paper is organized as follows: in Section 2 we formulate the estimation problem. The main theoretical results are stated in Section 3 and they are exemplified in Section 4. A summary discussion of the significance of the results closes the paper.

2. PROBLEM FORMULATION

Let $f(\cdot)$ be a function with the same domain and range space $U \subset \mathbb{R}$. The difference equation

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$s(n+1) = f(s(n))$, $n \in \mathbb{N}$, $s(0) \in U$ defines a *discrete time dynamical system* or *map*. Each *map orbit* (or generated *signal*) becomes defined for each *initial condition* s_0 and is denoted by $s(n, s_0)$ where $s(n, s_0) = f^n(s_0)$ with $f^n(\cdot)$ being the n -th successive application of $f(\cdot)$. For simplicity of notation, an orbit will be symbolized by $s(n)$ whenever s_0 is immaterial.

We seek an estimate of an orbit's initial condition s_0 from observations

$$s'(n) = s(n, s_0) + r(n), \quad (1)$$

$0 \leq n \leq N-1$, where $r(n)$ is gaussian zero mean white additive noise of variance σ_r^2 . The minimum mean square error (the CRLB) of any unbiased estimation method requires the computation of (under certain "regularity" conditions)

$$\text{mse}(\hat{\theta}) \geq \frac{1}{-E \left[\frac{\partial^2 \ln p(\mathbf{s}'; \theta)}{\partial \theta^2} \right]} \quad (2)$$

where $\mathbf{s}' = [s'(0), s'(1), \dots, s'(N-1)]^T$ is the vector of observations. The parameter of interest $\theta = s_0$ is statistically described by the likelihood function $p(\mathbf{s}'; \theta)$ (Kay, 1993).

Our objective is to obtain an expression for the CRLB as a function of statistical properties of the attractor. For this, it is worth reviewing what the Lyapunov number of a chaotic attractor is.

If $f(\cdot)$ is differentiable at each orbit point $s(n, s_0)$, the **orbit's Lyapunov number** is defined as:

$$L(s_0) = \lim_{N \rightarrow \infty} \left(\prod_{n=0}^{N-1} \left| \frac{df}{ds} \Big|_{s(n)} \right| \right)^{\frac{1}{N}} \quad (3)$$

if the limit in (3) exists. A non-periodic orbit is chaotic whenever $L(s_0) > 1$ and the larger its value, the faster is the orbit's split from other neighboring orbits with arbitrarily close initial conditions (Alligood, 1996).

Remark 1. In general, the chaotic orbits of a map converge to the same chaotic attractor, so the Lyapunov number of these orbits and of the attractor become identical (Alligood, 1996) leading to the common practice of calling this the Lyapunov number of the map.

3. 1-D MAP INITIAL CONDITION ESTIMATION CRLB

Theorem 1 establishes the CRLB for the problem of estimating the initial condition of an orbit as a function of the derivatives of $f(\cdot)$.

Theorem 1. The CRLB for estimating s_0 given \mathbf{r} in (1) and $f(\cdot)$ is given by:

$$\text{mse}(\hat{s}_0) \geq \frac{\sigma_r^2}{1 + \sum_{n=1}^{N-1} \left(\prod_{j=0}^{n-1} \frac{df}{ds} \Big|_{s(j)} \right)^2}. \quad (4)$$

Proof 1. Using the independence among the random variables $r(n)$ and (1), it follows that the likelihood function $p(\mathbf{s}'; s_0)$ equals

$$\frac{1}{(2\pi\sigma_r^2)^{\frac{N}{2}}} \exp \left[-\frac{1}{2\sigma_r^2} \sum_{n=0}^{N-1} (s'(n) - s(n))^2 \right], \quad (5)$$

which leads to

$$\frac{\partial^2 \ln p(\mathbf{s}'; s_0)}{\partial s_0^2} = \frac{1}{\sigma_r^2} \sum_{n=0}^{N-1} \left[(s'(n) - s(n)) \frac{\partial^2 s(n)}{\partial s_0^2} - \left(\frac{\partial s(n)}{\partial s_0} \right)^2 \right] \quad (6)$$

after differentiating its logarithm twice. Computing the expectation, we obtain

$$E \left[\frac{\partial^2 \ln p(\mathbf{s}'; s_0)}{\partial s_0^2} \right] = -\frac{1}{\sigma_r^2} \sum_{n=0}^{N-1} \left(\frac{\partial s(n)}{\partial s_0} \right)^2. \quad (7)$$

To arrive in Eq.(4) one must relate the derivatives of $f(\cdot)$ to those in (7). Using the chain rule and $s(n+1) = f(s(n))$ leads to

$$\frac{\partial s(n)}{\partial s_0} = \frac{df^n}{ds} \Big|_{s_0} = \begin{cases} \prod_{j=0}^{n-1} \frac{df}{ds} \Big|_{s(j)}, & n \geq 1 \\ 1, & n = 0 \end{cases}. \quad (8)$$

Substituting (8) into (7) leads to (4) in light of (2) and proves the theorem. \square

Theorem 1 holds for both chaotic or non-chaotic orbits, so that, in general, the CRLB for initial condition estimation depends on the very value s_0 being estimated. For chaotic orbits, however, this dependence becomes less pronounced when N grows as $s(n, s_0)$ tends to roam all over the attractor. This property is called *topological transitivity* (Devaney, 2003) and appears clearly in the next result.

Theorem 2. Under the same conditions of Theorem 1 and considering an orbit $s(n, s_0)$ the CRLB is given by:

$$\text{mse}(\hat{s}_0) \geq \sigma_r^2 \frac{L^2 - 1}{L^{2N} - 1}, \quad (9)$$

when $N \rightarrow \infty$ where $L \neq 1$ is the Lyapunov number of the attractor for which the orbit $s(n, s_0)$ converges.

Proof 2. For sufficiently large n , we can use the Lyapunov number definition (3) to approximately calculate

$$\left(\frac{\partial s(n)}{\partial s_0} \right)^2 = \left(\prod_{j=0}^{n-1} \frac{df}{ds} \Big|_{s(j)} \right)^2 \approx L^{2n}(s_0). \quad (10)$$

Thus, from (8) and from the fact that $L(s_0) \neq 1$,

$$\sum_{n=0}^{N-1} \left(\frac{\partial s(n)}{\partial s_0} \right)^2 \approx \sum_{n=0}^{N-1} L^{2n}(s_0) = \frac{L^{2N}(s_0) - 1}{L^2(s_0) - 1} \quad (11)$$

for N sufficiently large.

Hence as the orbit converges to an attractor whose Lyapunov number is L , (7) and (2) lead to (9) demonstrating the theorem. \square

Even for small values of N , (9) can be used as an approximation to the CRLB for chaotic orbits. This is so because L , is a geometrical mean of the derivatives of the map at the orbit points that replaces the true product of the derivatives at the first N points of the orbit. Due to the topological transitivity, this simplification leads to good results.

Furthermore (9) implies that the estimation error decreases exponentially with N . This result exposes quantitatively how the Lyapunov number of the attractor influences initial condition estimate error limits for large N . The larger L , the smaller the minimum mse of this estimator.

Sensitive dependence on initial conditions is characteristic of chaotic orbits. Thus an estimation error ϵ , implies that the error in computing $s(N-1)$ from the estimated value \hat{s}_0 and $f(\cdot)$ is $O(\max(s(N-1), \epsilon L^{N-1}))$. Hence, even if the initial condition estimate is relatively precise this does not imply a precise reconstruction of the orbit generated by s_0 .

A version of these theorems for piecewise linear maps was derived in (Papadopoulos and Wornell, 1993). This problem is also treated in different ways by (Abarbanel, 1996; Kay, 1995; Kay and Nagesha, 1995), yet none of these works provided general explicit formulae for the CRLB as a function of the map's Lyapunov number.

4. NUMERICAL EXAMPLE

In this section, we apply our results to the family of skew tent maps $s(n+1) = f_I(s(n))$ defined by

$$s(n+1) = \begin{cases} \frac{2}{\alpha+1}s(n) + \frac{1-\alpha}{\alpha+1}, & -1 < s(n) < \alpha \\ \frac{2}{\alpha-1}s(n) - \frac{\alpha+1}{\alpha-1}, & \alpha \leq s(n) < 1 \end{cases} \quad (12)$$

where $\{\alpha, s_0\} \subset (-1, 1)$. This map is shown in Figure 1a.

It can be shown (Kisel et al., 2001) that the Lyapunov number of a map of this family is given by:

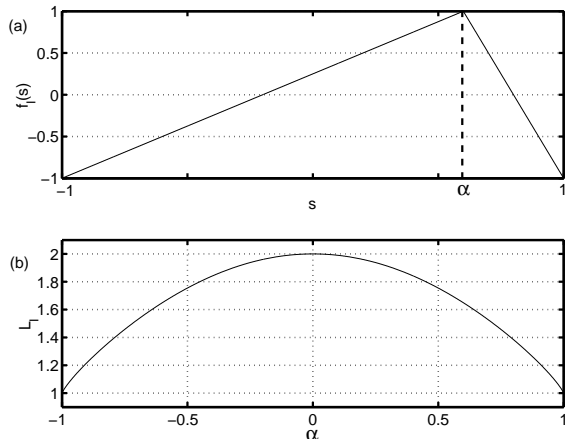


Fig. 1. (a) Skew tent map $f_I(s)$ and (b) its Lyapunov number as a function of α .

$$L_I = \left(\frac{2}{\alpha+1} \right)^{\frac{\alpha+1}{2}} \left(\frac{2}{1-\alpha} \right)^{\frac{1-\alpha}{2}}. \quad (13)$$

Figure 1b shows how the Lyapunov number L_I varies as a function of α .

The initial condition CRLB for $s_0 = 0$ is shown in Figure 2 as function of α for $\sigma_r = 1$ and different values of N . For each N two curves are plotted: a continuous one obtained via (4) and a dashed one obtained via the approximation (9). It is clear from these curves that (9) is really a good approximation for the CRLB even for small values of N , as stated.

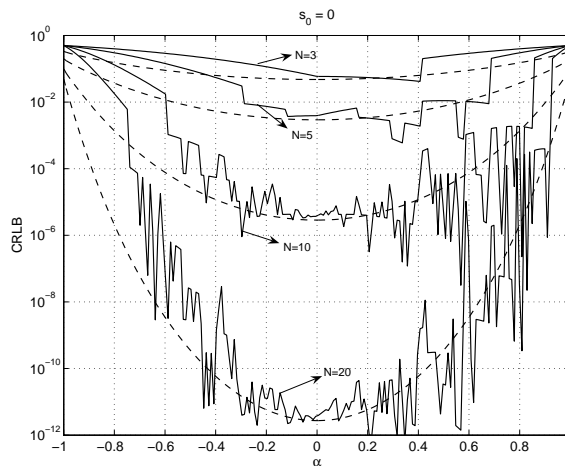


Fig. 2. CRLB on the estimation of the initial condition of the orbit $s_n(0)$ of $f_I(\cdot)$ as a function of α .

The smallest values of the CRLB, independently from the number of points used in the estimation, are in the vicinity of $\alpha = 0$, which provides the largest L_I , as suggested by Figure 1b. We have obtained similar curves for other values of s_0 .

When it comes to estimating the initial condition of an orbit, these results categorically indicate that, among the maps $f_I(\cdot)$, the best performance

is reached by the tent map $f_T(\cdot)$ obtained when $\alpha = 0$,

$$f_T(s) = 1 - 2|s|. \quad (14)$$

To verify our results we have implemented a maximum likelihood estimator (MLE) \hat{s}_0 (based on the one in (Papadopoulos and Wornell, 1993)) for the tent map $f_T(\cdot)$.

A known property of the MLE is its asymptotically unbiased and efficient character, i.e. the CRLB is reached for N sufficiently large (Kay, 1993).

To measure the MLE algorithm's performance, consider the *estimation gain* defined by

$$G = \frac{\sigma_r^2}{\text{mse}(\hat{s}(0))} = \frac{E \left[(s'(0) - s(0))^2 \right]}{E \left[(\hat{s}(0) - s(0))^2 \right]}. \quad (15)$$

Since the MLE is asymptotically unbiased, using $L = 2$, in (9) reduces it to

$$\text{mse}(\hat{s}_0) \geq \sigma_r^2 \frac{3}{4^N - 1} \quad (16)$$

and

$$G \leq \frac{4^N - 1}{3}. \quad (17)$$

The algorithm was tested for different N as a function of the signal-to-noise ratio:

$$SNR_{in} = \frac{\sum_{n=0}^{N-1} s(n)^2}{N\sigma_r^2}, \quad (18)$$

and the results are shown in Figure 3.

For each N value, two curves were plotted: a continuous one showing algorithm gain and a dashed one for (17) thereby confirming the validity of the theoretical limit.

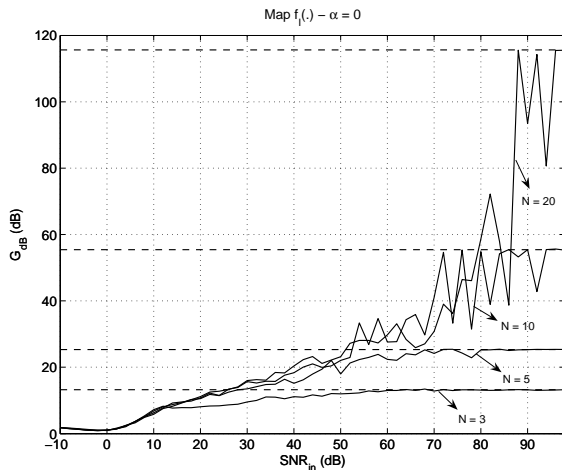


Fig. 3. Estimation gain of the MLE for the initial condition of the tent map $f_T(\cdot)$.

5. CONCLUSIONS

Theorems 1 and 2 are the main results of this paper and provide the CRLB for the estimation of s_0 under AWGN, leading, in principle, to a criterion for choosing which one-dimensional map might be best suited for chaotic communication system applications. The larger the Lyapunov number of a map, the smaller the CRLB of its efficient initial condition estimator and hence, the better its estimates. We stress that these theorems can be applied to any one-dimensional map with known derivative.

Another interesting conclusion from (9) is that semi-conjugative maps (Devaney, 2003) have the same CRLB performance for initial conditions for large N because they share the same Lyapunov number. For instance, the quadratic map $f_Q(s) = -2s^2 + 1$ for $s \in [-1, 1]$ and $f_T(s)$ perform identically for large N as $L_T = L_Q = 2$. When N is small, however, performance becomes a function of s_0 and more detailed analysis is necessary.

It possible to envision chaotic communication systems based on the estimation of the initial condition, either as an aid to detecting the transmitted information or for improving SNR prior to detection. For example, a very simple idea is to use N points of a tent map to transmit a binary symbol so that the information is coded by the initial condition using a randomly selected positive initial condition to transmit one symbol and a randomly selected negative one to transmit the other symbol. Systems like this are currently under research.

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