# The Impact of the Lyapunov Number on the Cramér-Rao Lower Bound for the Estimation of Chaotic Signals


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***Abstract*— This paper examines the problem of estimating initial conditions of noise-embedded chaotic signals generated by one-dimensional maps. A general expression for the Cramér-Rao lower bound is derived. When a large number of samples are used in the estimation, a relationship of this bound and the Lyapunov number of the attractor of the map is shown. These results can be used to choose the chaotic generator more suitable for applications on chaotic digital communication systems.**



***Index Terms*— Chaos, chaotic communications, spread spectrum communication, noise performance bounds.**


## I. Introduction

OVER the last ten years a large number of papers involving the application of chaotic signals in digital communications has appeared, e.g. [1] - [6]. In chaotic digital communications, the digital information is mapped directly to a wide-band chaotic signal. The principal difference between a chaotic carrier and a conventional periodic carrier is that the sample function for a given symbol is nonperiodic and is different from one symbol interval to the next.

Chaotic digital communication systems are spread spectrum systems. Thus, the interest in this systems lies in many desirable properties such as mitigation of multipath effects and the reducing of the transmitted power spectral density in order to minimize interference with other radio communications in the same frequency band [7].

As each chaotic system generates signals with different characteristics, a natural question is how to choose the most desirable chaotic signal for use in communication systems. While the answer still remains largely open [1], the important class of systems using piece-wise linear maps [8] is the most used to generate chaotic signals because of its simplicity. In its use, however, there is very little concern about the existence of an optimum chaos generating map that can improve performance.

In digital communication systems employing one-dimensional chaotic maps, often, the identification of the transmitted bit requires determining the initial condition $s_0$ of the chaotic dynamics associated with an observed information bit. Thus, in this paper we address the question of comparing generic one-dimensional chaos generating maps. We use the statistical criterium known as the Cramér-Rao lower bound (CRLB). We relate this performance bound to a known descriptor of the chaoticness of a map: the Lyapunov number. The CRLB sets a theoretical limit on the attainable precision in the estimation of $s_0$.

Indeed, as there are many well-known numerical techniques for estimating the Lyapunov number of a map [9], once established, this relationship may be the key in predicting the optimum noise performance of a communication system using this map as chaotic generator.

In Section II we formulate the estimation problem and review the concepts of Lyapunov number and Cramér-Rao lower bound. The main theoretical results are stated in Section III and they are exemplified in Section IV. Section V contains a summary of our conclusions.

## II. Problem Formulation

The problem of estimating the initial condition of a chaotic signal can be formulated as follows.

Consider the one-dimensional chaotic dynamical system

$$s_{n+1} = f(s_n) \tag{1}$$

where $\{s_0, s_1, \ldots, s_n, \ldots\}$ is the orbit generated by $f(.)$ with an initial condition $s_0$. This signal is corrupted by zero mean white gaussian noise $w_n$ with variance $\sigma_w^2$:

$$x_n = s_n + w_n \tag{2}$$

and its observed samples $x_n$ are available for $0 \leq n \leq N-1$.

Our goal is to compute the optimum performance bound that an unbiased estimator of the initial condition $s_0$ can attain by knowing the generating map $f(\cdot)$ and the observed sequence $\{x_0, x_1, \ldots, x_{N-1}\}$. In this paper, our optimality criterion is represented by the Cramér-Rao lower bound which we show to be

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a function of the Lyapunov number of the orbit traced from the initial condition $s_0$.

### A. Lyapunov number for one-dimensional maps

Consider the one-dimensional map $s_{n+1} = f(s_n)$. If $f(\cdot)$ is differentiable at the orbit points starting with initial condition $s_0$, we define this ***orbit's Lyapunov number*** $L(s_0)$ as:

$$L(s_0) = \lim_{i\to\infty}\left(\left|\frac{\partial f(s)}{\partial s}|_{s_0}\cdot\ldots\cdot\frac{\partial f(s)}{\partial s}|_{s_i}\right|\right)^{\frac{1}{i}} \quad (3)$$

if the limit in (3) exists.

The larger the Lyapunov number of an orbit, the faster is its split from other neighboring orbits [10]. This intuitively suggests that the larger the Lyapunov number, the easier should our estimates of the initial condition be. This result is confirmed rigorously in Section III.

### B. Cramér-Rao lower bound

The Cramér-Rao lower bound (CRLB) determines the smallest variance that an unbiased estimator can attain. Its knowledge in a certain problem allows answering whether a given unbiased estimator is efficient [11]. At the same time, it puts a cap on the least physically attainable variance by an unbiased estimator.

In the case of a signal depending on a single scalar parameter $\theta$, $s_n(\theta)$, corrupted by white gaussian noise $w_n$ with variance $\sigma_w^2$:

$$x_n = s_n(\theta) + w_n \qquad 0 \le n \le N-1, \quad (4)$$

the CRLB for $\theta$ is given by [11]

$$var(\hat{\theta}) \ge \frac{\sigma_w^2}{\sum_{n=0}^{N-1}\left(\frac{\partial s_n(\theta)}{\partial\theta}\right)^2}. \quad (5)$$

At this point, it is worth bearing in mind for latter comparison that a constant signal, $s_n = A$, in zero mean white gaussian noise with variance $\sigma_w^2$, has an associated CRLB in estimating $A$ given by

$$var(\hat{A}) \ge \frac{\sigma_w^2}{N} \quad (6)$$

when using $N$ observations as readily deducible from (5). This corresponds to a slow $O(1/N)$ decrease as $N \to \infty$.

## III. CRLB FOR ESTIMATING THE INITIAL CONDITION OF ONE-DIMENSIONAL CHAOTIC MAPS

To estimate $s_0$ from $\{x_0, x_1, \ldots, x_{N-1}\}$ as defined by (2), one can compute the CRLB using (5) as:

$$var(\hat{s_0}) \ge \frac{\sigma_w^2}{\sum_{n=0}^{N-1}\left(\frac{\partial s_n}{\partial s_0}\right)^2}. \quad (7)$$

Using the chain rule and (1) leads to

$$\frac{\partial s_n}{\partial s_0} = \left.\frac{\partial f(s)}{\partial s}\right|_{s_0}\cdot\left.\frac{\partial f(s)}{\partial s}\right|_{s_1}\cdot\ldots\cdot\left.\frac{\partial f(s)}{\partial s}\right|_{s_{n-1}}. \quad (8)$$

Hence,

$$var(\hat{s_0}) \ge \frac{\sigma_w^2}{\sum_{n=0}^{N-1}\left(\left.\frac{\partial f(s)}{\partial s}\right|_{s_0}\cdot\ldots\cdot\left.\frac{\partial f(s)}{\partial s}\right|_{s_{n-1}}\right)^2}, \quad (9)$$

which shows how to find the CRLB numerically when estimating $s_0$ from $N$ consecutive orbit samples under additive noise perturbation.

In general, $var(\hat{s_0})$ depends on $s_0$, the initial condition being estimated, i. e. the initial conditions do not enjoy uniform efficiency in their estimation. In the case of chaotic orbits, however, this nonuniformity becomes less pronounced as $N$ grows. This occurs do to the $f(\cdot)$'s topological transitiveness, or equivalently as orbits roam all over the attractor [12].

For a sufficiently large $n$, we can use (3) to approximately compute

$$\left(\left.\frac{\partial f(s)}{\partial s}\right|_{s_0}\cdot\ldots\cdot\left.\frac{\partial f(s)}{\partial s}\right|_{s_{n-1}}\right)^2 \approx L^{2n}(s_0), \quad (10)$$

which leads to

$$\sum_{n=0}^{N-1}\left(\frac{\partial s_n}{\partial s_0}\right)^2 \approx \sum_{n=0}^{N-1} L^{2n}(s_0) = \\ = \frac{L^{2N}(s_0) - 1}{L^2(s_0) - 1}. \quad (11)$$

by substitution into the denominator of (7) for a sufficient large $N$.

Hence, considering an orbit that converges to a chaotic attractor, we conclude that the CRLB of its initial condition estimation using $N \to \infty$ samples of this orbit corrupted by white gaussian noise is given by

$$var(\hat{s_0}) \ge \sigma_w^2\frac{L^2 - 1}{L^{2N} - 1}, \quad (12)$$

where $L$ is the Lyapunov number of the attractor and $\sigma_w^2$ is the variance of the noise $w_n$. Similar asymptotic exponential CRLB decay in a more particular case was reported in [13].

Equation (12) shows that the estimation error decreases exponentially with $N$ as a direct consequence of the information generation characteristic of chaotic systems. This result shows quantitatively how the Lyapunov number of the attractor influences the error limits on the initial condition estimation when $N$ is sufficiently large. The larger $L$, the smaller the minimal variance of this estimator as heuristically argued in Section II.A.

The exponential decay of the CRLB with $N$ is much faster than that involving the estimation of a constant signal which decreases only as $1/N$, as shown by (6). Theoretically, therefore, if unbiased estimators are available it should be possible to construct very accurate initial condition estimators.

The sensitive dependence on the initial condition is characteristic of the chaotic orbits. Thus, to obtain the initial condition with a relatively high precision does not imply the precise reconstruction of the orbit generated by $s_0$. If the initial condition's estimation error is $\epsilon$, then the error in computing $s_N$ using the estimated value $\hat{s_0}$ and $f(\cdot)$ will be $O\left(\max\left(s_N, \epsilon L^N\right)\right)$.

Larger Lyapunov numbers imply greater precision in $s_0$ estimates. However, this advantage is offset by the large error amplification when calculating subsequent orbit points. Thus, if our aim is to encode and recover information solely by our choice of $s_0$, then we should choose maps and initial conditions with large Lyapunov numbers but should not expect good recovery of the full orbit.

## IV. Examples

### A. Bernoulli shift map

Consider the Bernoulli shift map [7]:

$$s_{n+1} = f_B\left(s_n\right) = 2s_n \bmod 1, \tag{13}$$

defined on the interval $I = [0, 1]$.

This piecewise linear map, shown in Figure 1, has chaotic orbits with Lyapunov number $L_B = 2$. As the derivative of $f_B(.)$ is constant with respect to $s_n$, we can compute the CRLB directly from (12) for every $N > 0$:

$$var(\hat{s_0}) \geq \sigma_w^2 \frac{3}{4^N - 1}. \tag{14}$$

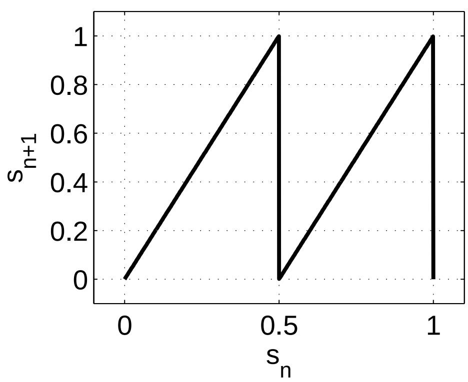


Fig. 1. Bernoulli shift map (13)

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Figure 2 shows the allied CRLB for initial condition estimation of (13) as $N$ grows ($\sigma_w^2 = 1$). For comparison, it shows the CRLB associated with estimating a constant (DC) value in noise according to (6).

### B. Logistic map

A popular one-dimensional chaotic map is

$$s_{n+1} = f_\ell\left(s_n\right) = 4s_n(1 - s_n), \tag{15}$$

the logistic map, one of whose orbits is exemplified in Figure 3.

In this case, the derivative of $f_\ell$ with respect to $s_n$ is not constant and (12) is valid only for $N \to \infty$. Hence, when $N$ is small, (9) must be used. The associated CRLB bound for various $s_0$ values is portrayed in Figure 4 using $N = 3$ samples and $\sigma_w^2 = 1$ showing the strong dependence of CRLB on the choice of the

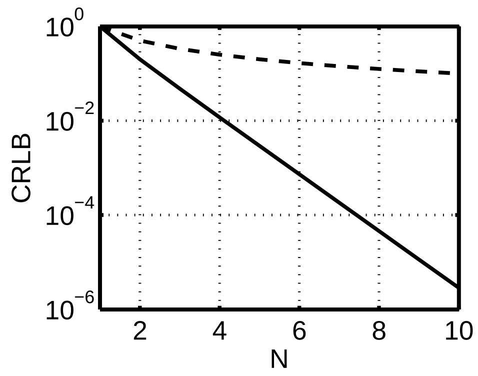


Fig. 2. CRLB for the estimation of the initial condition of the Bernoulli map (solid curve) and for a constant value (dashed curve) in additive white gaussian noise with $\sigma_w^2 = 1$

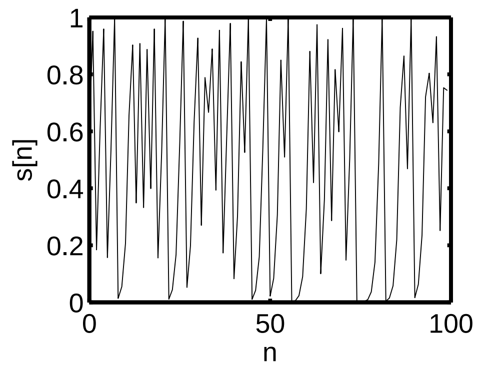


Fig. 3. Signal generated by the logistic map (15) with initial condition $s_0 = 0.61$.

initial condition and that the value $s_0 = 0.5$ is the least favorable choice in this case. This happens due to the nullity of the derivative of $f_\ell(\cdot)$ at this point. Thus, the orbits starting close to $s_0 = 0.5$ split more slowly from other close orbits hindering clear distinction of their initial conditions. The opposite occurs for $s_0$ close to the extremities of the interval $[0, 1]$ where the absolute value of the derivative of $f_\ell(.)$ is maximum.

Figure 5 portrays CRLB dependence on $s_0$ for different values of $N$ and illustrates our theoretical discussion concerning the decreased dependence on $s_0$ when observation times are longer.

The Lyapunov number of an orbit of the logistic

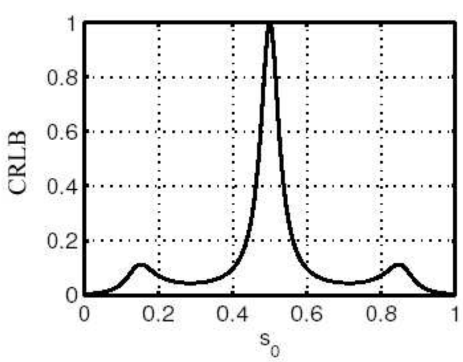


Fig. 4. CRLB for the estimation of $s_0$ for $N = 3$ samples.

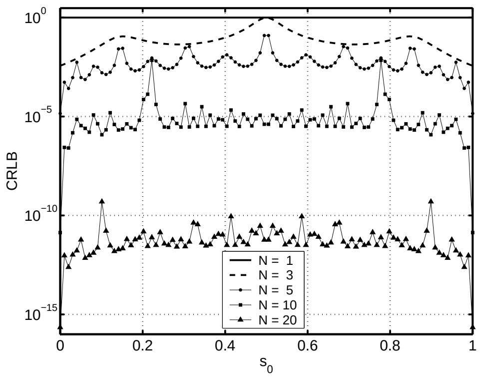


Fig. 5. CRLB for the estimation of $s_0$ for the map $f_\ell$ and various values of $N$.

map with initial condition on the interval $[0, 1]$, excluding a null measure set, is $L_\ell = 2$ [10]. Therefore, using (12), the same numerical result as for the Bernoulli map (14) is obtained for $N \to \infty$.

As a numerical illustration, using (14) when $N = 10$ and $\sigma_w^2 = 1$, leads to $var(\hat{s_0}) \approx 10^{-6}$ in accord with Figure 5, thereby validating the asymptotic approximation even for relatively small $N$.

## V. Conclusions

Equation (12) shows the intimate relationship between the statistical properties of the chaotic attractor (Lyapunov number) and the performance of a would be efficient initial condition estimator of noise-embedded chaotic signals given by the CRLB when sufficiently many samples are used.

This result, in principle, provides a criterion for choosing the best one-dimensional map to be used to generate a chaotic signal from which to recover the initial condition under additive noise. Hence, if it is possible to find unbiased estimators, the larger the Lyapunov number of a map, the smaller the CRLB of its efficient initial condition estimator and hence, the better its estimates.

Another interesting conclusion from Equation (12) is that semi-conjugative maps [10] have the same CRLB performance for initial conditions for large $N$ because they share the same Lyapunov number. For instance, $f_B(\cdot)$ and $f_\ell(\cdot)$ perform identically for large $N$ under this criterion as $L = 2$ for both. When $N$ is small, however, performance depends on the initial condition $s_0$.

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