The 7th International Telecommunications Symposium (ITS 2010)

Performance comparison between single-carrier and OFDM using the cutoff rate

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***Abstract*— This article aims to establish a performance comparison between the Single Carrier with Cyclic Prefix (SCCP) and the Orthogonal Frequency Division Multiplexing (OFDM) systems in terms of the cutoff rate. We consider both linearly equalized SCCP (LE-SCCP) and the SCCP equalized with a decision feedback equalizer (DFE-SCCP). A theoretical approach to the problem is proposed and simulation results showing the cutoff rate for several coding rates are also presented. Still, the systems BLER are evaluated and its dependency on the coding rate are analyzed and compared to the behavior shown by the cutoff rate analysis.**

Keywords*— *OFDM, frequency domain equalization, single-carrier, DFE, cutoff rate, channel coding.

I. Introduction

The orthogonal frequency division multiplexing (OFDM) is a popular transmission technique, where digital signals are transmitted over orthogonal subcarriers. As long as a sufficient cyclic prefix (CP) is appended in the transmission, the received signal can be easily equalized using a frequency domain one-tap equalizer. On the other hand, this simple equalization structure is not restricted to the OFDM system. If the same CP solution is applied to the single-carrier (SC) system, we can also use the same frequency domain one-tap equalizer structure. We refer to this scheme as SCCP [1], [2], [3].

The computational complexity involved in the implementation of both OFDM and SCCP can be shown to be approximately the same [1]. This fact leads to several comparisons between these two schemes [1], [2], [4], [5], [6]. Some are restricted to the uncoded case [5], [6] and, in this context, theoretical bounds are derived. Other comparisons consider the coded scenario restricting the analysis to Monte Carlo simulation [1], [2], [4].

The uncoded case comparisons have shown that the OFDM presents a poor performance when compared to the SCCP system [5], [6]. This is due to the fact that, in the uncoded scenario and in the absence of channel state information at the transmitter side, the OFDM system is not able to explore the channel frequency diversity. This is not true for the SCCP or the coded OFDM systems, where the information symbols are spread among all subcarriers.

The comparisons concerning a coded scenario [1], [2], [4] have shown, through simulations, that the OFDM can have a performance equal to or even better than the SCCP depending on the chosen modulation and coding rate. Particularly,

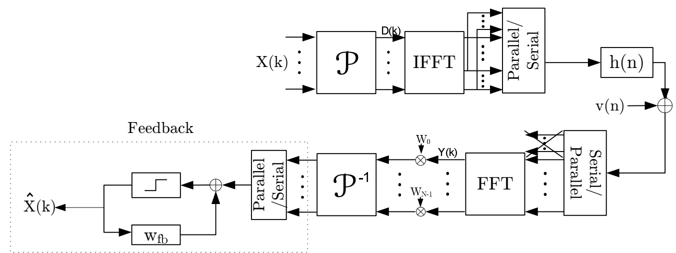

Fig. 1. System model

the effect of the coding rate was analytically investigated in [7]. A lower bound of the channel capacity, the cutoff rate parameter [8], was analyzed and it was shown that for high coding rates the linearly equalized single-carrier with cyclic prefix (LE-SCCP) outperforms the OFDM system and that the performance gap decreases with decreasing coding rate. This fact is corroborated by simulation results presented in the literature [1], [2], [4].

However, the analysis provided in [7] is restricted to a two-path Rayleigh channel model and it does not provide any insight, in terms of the cutoff rate, about the performance of the SCCP using a decision feedback equalizer (DFE), which is well-known for its superior performance when compared to a linear equalizer (LE) [9]. In this paper, we analyze the DFE-SCCP using the cutoff rate and, through a convex analysis framework, we are able to predict the performance difference between the OFDM and DFE-SCCP systems for any given channel configuration.

The rest of this paper is organized as follows. In section II, we present the system model employed in the analysis. Section III presents the performance comparison in terms of the cutoff rate and the convex analysis that allows us to generalize the conclusions for a given channel configuration. In section IV, we present some simulation results and finally, in section V, conclusions are stated.

II. System Model

The Fig. 1 shows a unified system model to describe both OFDM and SCCP systems. This universal approach based on linear precoding was proposed in [6].

The linear precoding matrix $\mathcal{P}$ is the only factor that determines the system represented by the model. In the OFDM case, the transmitted symbols are generated from the IFFT (inverse of the fast Fourier transform) of the data vector $\mathbf{X}$, thus the data vector is not pre-processed and the transformation $\mathcal{P}$ is replaced by the identity matrix. On the other hand, in the SCCP scheme, the symbol vector itself is transmitted. This is achieved by making the precoding matrix $\mathcal{P}$ equal to the Fourier matrix.

The modulated symbols and the appended CP are transmitted over a channel with impulse response $\{h(n)\}$ and corrupted with additive white Gaussian noise (AWGN) $v(n)$. The CP is removed from the received signal and the one-tap equalization is carried out in the frequency domain. Finally, the linear decoding is achieved by multiplying the equalizer output by the inverse of the precoding matrix.

Concerning the OFDM transmission scheme, the received symbols are free from intersymbol interference (ISI). In such a case, the one-tap equalizer $W(k)$ provides a phase and magnitude correction, which is equivalent to obtain a ML estimation of the received symbol. In contrast, the ML detection is prohibitively complex for the SCCP and usually linear or decision feedback equalization in the frequency domain is used. This frequency domain approach leads to a similar computational complexity compared to the OFDM. In the SCCP, the equalization criterion should be carefully chosen. The zero-forcing (ZF) criterion eliminates the ISI, but it can lead to a prohibitive noise enhancement in deep fading channels. In such a situation, the minimum mean square error (MMSE) criterion is more suitable. For this reason, we will adopt the latter.

If the SCCP system is equalized with a DFE, a feedback filter should be appended in the receiver. As illustrated in Fig. 1, the feedforward filtering is accomplished in the frequency domain by the one-tap equalizer and the feedback filtering is accomplished in the time domain. The feedback and feedforward filters coefficients can be calculated as shown in [3]. The number of DFE feedback coefficients is equal to the channel length minus one [10]. We also assume perfect feedback decisions in order to make the mathematical analysis more feasible.

III. Performance Comparison

In this section, we will present the cutoff rate expressions for the OFDM and the SCCP systems [7] and, later, we provide the convex analysis that allow us to generalize the behavior of the OFDM and SCCP with regard to the coding rate for any channel.

Firstly, let us define the cutoff rate for the M-ary modulation and AWGN scenario [8]:

$$R_0 = -\log_2\left(\frac{1}{M^2}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}\mathbb{P}\{x_l \mapsto x_m\}\right) \tag{1}$$

where $\mathbb{P}\{x_l \mapsto x_m\}$ represents the pairwise error probability, *i.e.*, the probability of decoding a symbol as x_m given that x_l was transmitted.

The pairwise error probability is given by:

$$\mathbb{P}\{x_l \mapsto x_m\} = Q\left(\sqrt{\frac{\|x_l - x_m\|^2}{2\sigma_v^2}}\right) \tag{2}$$

where σ_v^2 represents the noise power.

Applying the Chernoff bound, the cutoff rate can be approximated by:

$$R_{0,AWGN} = -\log_2\left(\frac{1}{M^2}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}\exp\left\{-\frac{\|x_l - x_m\|^2}{4\sigma_v^2}\right\}\right) \tag{3}$$

The metric $\|x_l - x_m\|$ represents the euclidian distance between the symbols x_l and x_m. Defining:

$$A_{l,m} = \frac{\|x_l - x_m\|^2}{4\sigma_x^2} \tag{4}$$

where σ_x^2 is the symbol power, we can rewrite the cutoff rate as:

$$R_{0,AWGN} = -\log_2\left(\frac{1}{M^2}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}\exp\left\{-A_{l,m}\gamma\right\}\right) \tag{5}$$

where $\gamma = \frac{\sigma_x^2}{\sigma_v^2}$.

Considering now that the symbols are transmitted over a frequency selective channel with frequency response $H(f)$, we can express the cutoff rate of the OFDM as follows [7]:

$$R_{0,OFDM} = -\log_2\left(\frac{1}{M^2}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}\mathbb{E}\left\{\exp\left(-\gamma A_{l,m}|H(f)|^2\right)\right\}\right) \tag{6}$$

We can now invoke ergodicity to finally express the OFDM cutoff rate as:

$$R_{0,OFDM} = -\log_2\left(\frac{1}{M^2}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}\frac{1}{N}\sum_{k=0}^{N-1}\exp\left(-\gamma A_{l,m}|H_k|^2\right)\right) \tag{7}$$

where N is the number of subcarriers in the OFDM system and the symbol block length in the SCCP case.

Concerning the SCCP system, we can consider that any residual intersymbol interference after equalization can be modeled as a Gaussian random variable. Under this assumption, the cutoff rate of the SCCP system is given by:

$$R_{0,SCCP} = -\log_2\left(\frac{1}{M^2}\sum_{l=0}^{M-1}\sum_{m=0}^{M-1}\exp\left\{-A_{l,m}\mathrm{SNR}\right\}\right) \tag{8}$$

in which the SNR is the signal to noise ratio at the equalizer output.

If the SCCP signal is linearly equalized, the SNR is given by [9]:

$$\mathrm{SNR}_{\mathrm{LE}} = \frac{\sum_{k=0}^{N-1}\gamma|H_k|^2\Big/\left(1+\gamma|H_k|^2\right)}{\sum_{k=0}^{N-1}1\Big/\left(1+\gamma|H_k|^2\right)} \tag{9}$$

Using the harmonic mean operator, the above equation can be expressed as:

$$\mathrm{SNR}_{\mathrm{LE}} = \mathrm{harmean}\left\{1+\gamma|\mathbf{H}|^2\right\} - 1 \tag{10}$$

Conversely, if the signal is equalized with a perfect DFE, *i.e.* a DFE without error propagation, the SNR can be expressed as [11]:

$$\mathrm{SNR}_{\mathrm{DFE}} = \exp\left\{\frac{1}{N}\sum_{k=0}^{N-1}\log\left(1+\gamma|H_k|^2\right)\right\}-1 \tag{11}$$

This expression can also be rewritten in terms of the geometric mean operator:

$$\mathrm{SNR}_{\mathrm{DFE}} = \mathrm{geomean}\left\{1+\gamma|\mathbf{H}|^2\right\}-1 \tag{12}$$

It is known that the geometric mean is greater or equal the harmonic mean. Thus, we can infer from (10) and (12) that the SNR in the perfect DFE output is greater or equal to the SNR in the LE output. Thus, from (8), we can conclude that the cutoff rate associated to the DFE is greater or equal to the cutoff rate associated to the LE.

Defining the following relations:

$$\zeta_{OFDM} = \frac{1}{N}\sum_{k=0}^{N-1}\exp\left(-\gamma A_{l,m}|H_k|^2\right) \tag{13}$$

$$\zeta_{LE} = \exp\left\{-\mathrm{SNR}_{\mathrm{LE}}A_{l,m}\right\} \tag{14}$$

$$\zeta_{DFE} = \exp\left\{-\mathrm{SNR}_{\mathrm{DFE}}A_{l,m}\right\} \tag{15}$$

the eqs. (7) and (8) reveal that the monotonic behavior of the logarithmic function allows us to compare the cutoff rates by comparing the functions ζ_{OFDM}, ζ_{LE} and ζ_{DFE}. In such a case that $\zeta_a < \zeta_b \rightarrow R_{0,a} > R_{0,b}$. In order to compare the expressions in (13), (14) and (15), we define the following functions:

$$\theta(x) = \frac{1}{1+x} \tag{16}$$

$$\phi(x) = \log(x+1) \tag{17}$$

$$\tau(x) = \exp\left(-A_{l,m}x\right) \tag{18}$$

Finally, (13), (14) and (15) can be rewritten as:

$$\zeta_{OFDM} = \frac{1}{N}\sum_{k=0}^{N-1}\tau\left(\gamma|H_k|^2\right) \tag{19}$$

$$\zeta_{LE} = \tau\left(\theta^{-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}\theta\left(\gamma|H_k|^2\right)\right)\right) \tag{20}$$

$$\zeta_{DFE} = \tau\left(\phi^{-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}\phi\left(\gamma|H_k|^2\right)\right)\right) \tag{21}$$

where the function $\phi^{-1}(x)$ and $\theta^{-1}(x)$ are the inverse functions of $\phi(x)$ and $\theta(x)$, which are given by $\phi^{-1}(x) = \exp(x)-1$ and $\theta^{-1}(x) = \frac{1}{x}-1$.

Written is such form, (19), (20) and (21) allow the use of a convex analysis in order to establish a comparison between the OFDM and SCCP for any given configuration. In addition, we divide the comparison between the OFDM and SCCP in two different contexts: a) comparison between the OFDM and the LE-SCCP and b) comparison between the OFDM and the DFE-SCCP.

In order to accomplish the first comparison, we also define:

$$\rho(x) = \tau\left(\theta^{-1}(x)\right) \tag{22}$$

This function allow us to express (19) and (20) as:

$$\zeta_{OFDM} = \frac{1}{N}\sum_{k=0}^{N-1}\rho(x_k) \tag{23}$$

$$\zeta_{LE} = \rho\left(\frac{1}{N}\sum_{k=0}^{N-1}x_k\right) \tag{24}$$

where

$$x_k = \theta\left(\gamma|H_k|^2\right) \tag{25}$$

The Jensen inequality guarantees that if the function ρ is convex, $\rho\left(\frac{1}{N}\sum_{k=0}^{N-1}x_k\right) \leq \frac{1}{N}\sum_{k=0}^{N-1}\rho(x_k)$. Hence, the convexity of ρ implies $\zeta_{LE} \leq \zeta_{OFDM}$ that is equivalent to state that $R_{0,\mathrm{LE}} \geq R_{0,\mathrm{OFDM}}$.

By definition, a function is convex if and only if its second derivative is non-negative. The second derivative of the function $\rho(x)$ is given by:

$$\frac{d^2}{dx^2}\rho(x) = A_{l,m}\frac{\rho(x)}{x^3}\left(\frac{A_{l,m}}{x}-2\right) \tag{26}$$

and as the function $\rho(x)$ is non-negative, the sign of $\frac{d^2}{dx^2}\rho(x)$ is determined by the function $a(x)$, given by:

$$a(x) = \frac{A_{l,m}}{x}-2 \tag{27}$$

However, (23) reveals that the x values that are important to the cutoff rate calculation are given by (25). Calculating the function $a(x)$ at this values, we have that:

$$a(x_k) = A_{l,m}\left(\gamma|H_k|^2+1\right)-2 \tag{28}$$

Moreover, considering a coding rate R, if we express γ in terms of the relation E_b/N_o, we have that:

$$a(x_k) = A_{l,m}\left(R\log_2(M)\frac{E_b}{N_o}|H_k|^2+1\right)-2 \tag{29}$$

In order to assure that $\zeta_{LE} \leq \zeta_{OFDM}$, we have to guarantee that $a(x_k) \geq 0$. For a fixed E_b/N_o, the channel coefficients need to guarantee that (29) is greater than zero and, as a consequence, $\zeta_{LE} \leq \zeta_{OFDM}$. Thus, for higher coding rates, it is possible to be less restrictive with regard to the channel coefficients to guarantee the positiveness of (29). This fact points that increasing coding rates represents a more favorable scenario for the LE-SCCP system.

We also can establish a comparison between the OFDM and the DFE-SCCP system in a similar way. In order to do so, we define the function:

$$\xi(y) = \tau\left(\phi^{-1}(y)\right) \tag{30}$$

Hence, (19) and (21) can be rewritten as:

$$\zeta_{OFDM} = \frac{1}{N}\sum_{k=0}^{N-1}\xi\left(y_k\right) \tag{31}$$

$$\zeta_{DFE} = \xi\left(\frac{1}{N}\sum_{k=0}^{N-1}y_k\right) \tag{32}$$

where

$$y_k = \phi\left(\gamma|H_k|^2\right) \tag{33}$$

We analyze the second derivative of the function $\xi(y)$ in order to verify its convexity:

$$\frac{d^2}{dy^2}\xi(y) = A_{l,m}\exp(y)\left(A_{l,m}\exp(y)-1\right)\xi(y) \tag{34}$$

The sign of $\frac{d^2}{dy^2}\xi(y)$ is determined by $b(y)$, given by:

$$b(y) = A_{l,m}\exp(y)-1 \tag{35}$$

and from (31) we can infer that y values that are used in the calculation of the cutoff rate are given by (33). Calculating the function $b(y)$ in this values, we have that:

$$b\left(y_k\right) = A_{l,m}\left(\gamma|H_k|^2+1\right)-1 \tag{36}$$

In terms of the relation E_b/N_o, we have that:

$$b\left(y_k\right) = A_{l,m}\left(R\log_2(M)\frac{E_b}{N_o}|H_k|^2+1\right)-1 \tag{37}$$

Therefore, we have a situation similar to that one presented in the comparison between the OFDM and the LE-SCCP system. The main difference is that the $b\left(y_k\right) > a\left(x_k\right)$. Hence, given a fixed E_b/N_o and coding rate, the channel class for which (37) is greater than zero is broader than that one which guarantees the positiveness of (29).

IV. Simulation Results

In this section we evaluate the cutoff rate of the analyzed systems when the transmission is accomplished in a frequency selective channel with the following transfer function:

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} \tag{38}$$

where the coefficients $h_k,\ k=0,1,2$ are Rayleigh variables with zero mean and variance given by $\sigma_h^2 = 1/3$.

We consider QPSK modulation and coding rates $R = 1/2, 3/4, 9/10$. The espectral efficiency associated to each coding rate is given by:

$$\eta_1 = R_1\log_2(M) = 1$$
$$\eta_2 = R_2\log_2(M) = 1.5$$
$$\eta_2 = R_3\log_2(M) = 1.8$$

For each coding rate, we evaluate the probability that the transmitted espectral efficiency is above the cutoff rate. We will refer to this probability as the cutoff rate outage probability. The cutoff rate outage probability is estimated through Monte Carlo simulation and the results are shown

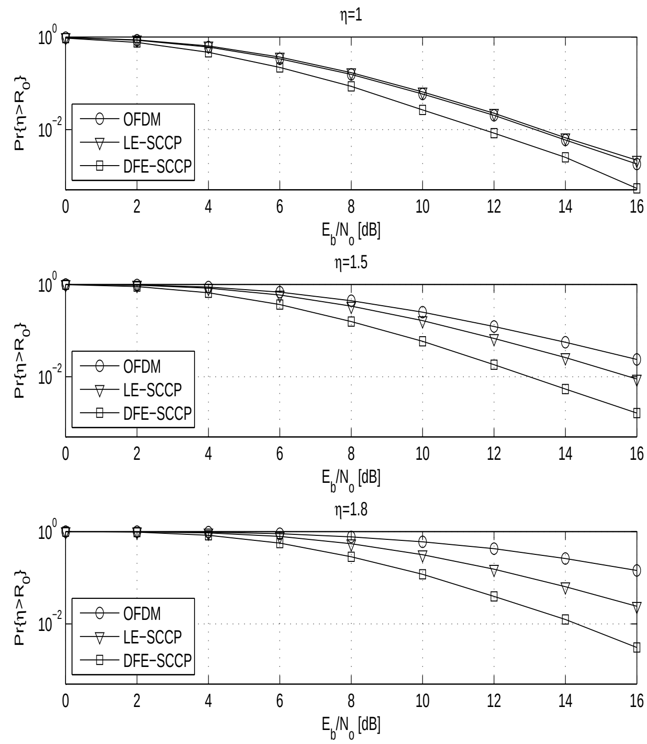

Fig. 2. Cutoff rate outage for QPSK modulation and channel with transfer function given by eq. (38).

in Fig. 2. From Fig. 2, we observe a degradation in the performance of the OFDM in comparison to the SCCP with increasing coding rate. For $\eta = 1$, the cutoff rate outage probability associated to the OFDM system is slightly lower than the one associated to the LE-SCCP. For higher coding rate, the LE-SCCP outperforms the OFDM. Concerning the DFE-SCCP, the OFDM presents an inferior performance for all simulated espectral efficiencies. This degradation in the OFDM performance with increasing coding rate is not unexpected, since it is known that the OFDM can not exploit the channel diversity in an uncoded scenario. Actually, in terms of BER, it can be proved that, concerning the QPSK modulation, the LE-SCCP outperforms the OFDM for any channel condition in the uncoded case [6].

In order to verify the behavior of the systems as a function of the coding rate in a more realistic scenario, *i.e.* when convolutional coding is considered instead of random coding, we analyze the block error rate (BLER) obtained when the transmission is accomplished with the convolutional error correcting code with generator polynomial $[133\ 171]_{\text{octal}}$ and coding rate $R = 1/2$, as well as the $R = 3/4$ code obtained from its punctering. The block length in the implemented system is five hundred twelve information symbols in the SCCP system or five hundred twelve subcarriers in the OFDM system. As pointed out by [12], the system behavior is highly dependent on the interleaving pattern, particularly for the OFDM. Due to this fact, several interleaving patterns

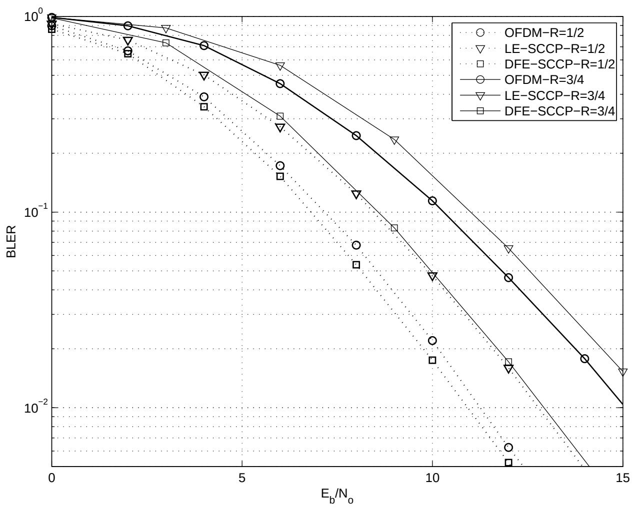

Fig. 3. BLER for QPSK modulation and channel with transfer function given by eq. (38). Error correcting code with generator polinomial $[133\ 171]_{\mathrm{octal}}$, coding rates $R = 1/2$ and $R = 3/4$

possibilities were tested and the block rectangular bit interleaver with $m = 8$ rows and $n = 256$ columns, where the interleaving is accomplished filling the interleaver matrix through its lines and reading through its columns, has been chosen due to its good performance for both systems. Fig. 3 shows the systems BLER for both coding rates.

From Fig. 3, we note that for the coding rate $R = 1/2$, the BLER of the OFDM is comparable to the BLER of the DFE-SCCP, being slightly higher. The LE-SCCP, on the other hand, presents a considerable higher BLER than those from OFDM and DFE-SCCP. For the coding rate $R = 3/4$, we can observe that the BLER provided by the OFDM is comparable to the one provided by the LE-SCCP, while the DFE-SCCP presents a far superior performance than the other two systems. This fact shows, once more, the dependency of the OFDM performance on the coding rate.

When comparing the cutoff rate analysis and BLER simulations, both show the same behavior with regard to the coding rate: with increasing coding rate, there is a performance degradation of the OFDM system compared to the SCCP. Nevertheless, it is worth noting that the cutoff rate analysis seems to be pessimistic about the OFDM performance when compared to the BLER simulations. As an example, considering a coding rate $R = 1/2$, the OFDM presents a performance similar to the LE-SCCP in terms of cutoff rate, while it presents a performance close to the DFE-SCCP in terms of BLER.

In addition, it should be emphasized that the DFE-SCCP analysis was not carried out considering the error propagation effect. Even though, the DFE-SCCP presents a superior performance when compared to the OFDM in the analyzed scenario, the performance degradation introduced by the error propagation might lead to situations where the OFDM surpasses the DFE-SCCP.

V. Conclusion

In this article, we have established a performance comparison between the OFDM, the LE-SCCP and DFE-SCCP systems in terms of cutoff rate and BLER. We have shown a degradation on the OFDM performance, concerning both cutoff rate and BLER, due to the increasing of the system coding rate. An analytical treatment exploring a convex analysis framework was provided and the predicted behavior pointed out by the theoretical analysis were confirmed by means of simulations.

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