Minimum Rates Scheduling for MIMO OFDM Broadcast Channels

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Abstract—In this paper we study the multiple input multiple output (MIMO) orthogonal frequency division multiplexing (OFDM) Gaussian Broadcast Channel (BC). Several fundamental problems are considered: The maximization of a weighted sum of rates and the dual minimization of sum power subject to rate requirements. Further we study the combined problem of weighted rate sum maximization under minimum rate requirements. To show the connections among them, all problems are embedded into an enhanced convex set of rate-power tuples $(R, P)$. This not only gives insights into the structure of the MIMO OFDM BC capacity region, but also motivates algorithms exploiting these properties.

I. INTRODUCTION

It was not until recently that the capacity region of the MIMO Gaussian BC was completely solved. In the line of work [11, 12, 13] it was shown that the MIMO BC capacity region in fact equals the MIMO BC dirty-paper-coding region and thus is equivalent to the capacity region of a dual MIMO multiple access channel (MAC) with hermitian transposed channels. These results are of fundamental interest, since they allow to carry out any analysis and optimization of the MIMO BC in the dual MAC and to carry over the results to the downlink via the corresponding duality relations [1]. Based on the established duality, some important resource allocation results can be derived:

- It is known that the boundary (i.e. the efficient points) of the capacity region can be parameterized by a weighted rate sum-maximization over the capacity region. Further, an algorithm for solving this problem is presented in [6].

II. SYSTEM MODEL

Consider a frequency selective MIMO OFDM Gaussian BC with $K$ subcarriers, a base station equipped with $U$ antennas and $M$ mobiles each owning antennas. In the following the set of users will be denoted as $\mathcal{M} = \{1, \ldots, M\}$. Assume a block fading process with independent and identical fading from block to block and let the fixed channel realization of user $m \in \mathcal{M}$ on subcarrier $k$ during a time block be $H_{m,k} \in \mathbb{C}^{V_m \times U}$ where $[H_{m,k}]_{i,j}$ is the complex channel coefficient between antenna $j$ of the base station and antenna $i$. Then the signal received by user $m$ on subcarrier $k$ is given by

$$y_{m,k} = H_{m,k}x_k + n_{m,k} \quad k = 1, \ldots, K. \quad (1)$$

with $x_k \in \mathbb{C}^{U \times 1}$ and $y_{m,k} \in \mathbb{C}^{V_m \times 1}$ being the transmitted and received signals, respectively. Here $n_{m,k} \in \mathbb{C}^{V_m \times 1}$ denotes the white Gaussian receiver noise with $\mathbb{E}\{n_{m,k}n_{m,k}^H\} = I_{V_m}$ with $I_n$ being the identity matrix of dimension $n$.

Now consider the following multiple access channel

$$r_k = \sum_{m=1}^{M} H_{m,k}s_{m,k} + z_k \quad k = 1, \ldots, K \quad (2)$$

The remainder of the paper is organized as follows. Section II introduces the system model. In Section III the problems are stated and basic properties are derived. Subsequently, in Section IV and V the sum power minimization and the minimum rates problem are studied, respectively. We conclude with Section VI.
with \( r_k \in C^U, s_{m,k} \in C^{V_0} \) and (normalized) additive white Gaussian noise \( \mathbb{E}\{z_kz_k^H\} = I_f \). This system is commonly called the dual MAC to the BC in (1) and it is known that their capacity regions under a common sum power constraint \( P \) coincide:

Stacking the channel matrices into a single matrix \( H = [H_{1,1}, \ldots, H_{M,K}]^T \), we have from [3]

\[
C_{BC}(H, P) \equiv C_{MAC}(H^H, P)
\]

where the capacity region of the Gaussian MIMO OFDM MAC is given by

\[
C_{MAC}(H, P) = \bigcup_{\sum_m, k \text{ tr}(Q_{m,k}) \leq P} \left\{ R : R(I) \leq f_{H,Q}(I), \forall I \subseteq M \right\}. \tag{4}
\]

with \( R(I) := \sum_{m \in I} R_m \) and

\[
f_{H,Q}(I) := \sum_{k=1}^K \log \det \left( I + \sum_{m \in I} H_{m,k} Q_{m,k} H_{m,k}^H \right) \tag{5}
\]

being a rank function. Here, \( A \succeq 0 \) means that \( A \) is a positive semidefinite matrix, \( \text{tr}(\cdot) \) denotes the trace operator and \( Q_{m,k} = \mathbb{E}\{s_{m,k} s_{m,k}^H\} \) is the covariance matrix of the \( m \)th user’s signal on subcarrier \( k \) stacked to the matrix \( Q = [Q_{1,1}, \ldots, Q_{M,K}]^T \) for brevity.

Any rate tuple \( R \in C_{BC}(H, P) \) achievable in the MIMO OFDM BC is also achievable in the dual MAC and vice versa. Further, the transformation laws relating the covariance matrices achieving the same rate tuple in MAC and BC are known [1]. Note that the characterization of the dual MIMO OFDM MAC in (4) involves the polymatroidal structure of elementary capacity regions - i.e. capacity regions for a fixed set of covariance matrices - which can be exploited in multiple ways. Thus we can focus on the dual MAC in the following and the subscripts \('MAC\) and \('BC\) will be omitted.

## III. PROBLEM STATEMENT AND BASIC PROPERTIES

The fundamental problem of maximizing a weighted sum of rates yields a characterization of the boundary of \( C(H, P) \) and was studied in [13] for the MIMO OFDM BC:

\[ \text{Problem 1:} \]

\[
\begin{aligned}
\max_{\mu} & \quad \mu^T\bar{R} \\
\text{subj. to} & \quad R \in C(H, P) \\
\end{aligned} \tag{6}
\]

Here \( \mu = [\mu_1, \ldots, \mu_M]^T \) is the vector of individual weights. Problem 1 can be turned into a convex problem with trace constraints via the following well known Lemma.

**Lemma 1:** The solution to Problem 1 is equivalent to solving

\[
\max_{S_{\pi_n}(H,P)} \mu^T\bar{R} \tag{7}
\]

where the set \( S_{\pi_n}(H,P) \) denotes the region achievable by one specific decoding order

\[
S_{\pi_n}(H, P) = \bigcup_{\sum_{m,k} \text{ tr}(Q_{m,k}) \leq P} \left\{ R : R_{\pi_n(m)} \leq f_{H,Q}(\{\pi_n(1), \ldots, \pi_n(m-1)\}) \right\}, \tag{8}
\]

where \( \pi_n \in \Pi \) is a permutation sorting the \( \mu_m \) in decreasing order:

\[
\mu_{\pi_n(1)} \geq \mu_{\pi_n(2)} \geq \ldots \geq \mu_{\pi_n(M)}. \tag{9}
\]

The proof exploits the polymatroidal structure of elementary capacity regions.

We want to consider two different important resource allocation strategies, which contrast with the pure weighted rate sum-maximization. Although this strategy is known to be stability-optimal in the sense of non-evanescence of queues [5], it might penalize users with a low queue but strict Quality of Service demands. Thus, we first consider the problem of finding the optimal transmit strategy and therefore the minimum transmit power for achieving a set of required rates \( \bar{R} = [\bar{R}_1, \ldots, \bar{R}_M]^T \).

**This problem can be formulated as:**

\[
\begin{aligned}
\min_{P} & \quad P \\
\text{subj. to} & \quad \bar{R} \in C(H, P) \\
\end{aligned} \tag{10}
\]

Geometrically speaking, we search for the smallest possible capacity region still containing the desired rate vector \( \bar{R} \).

If the sum power is limited to a fixed budget \( P \) in each fading state and stability issues can not be neglected completely, both perspectives can be combined. This leads to the problem of weighted rate sum-maximization under given minimum rates \( \bar{R} \) (the inequality refers to component-wise greater or equal):

**Problem 3:**

\[
\begin{aligned}
\max_{\mu} & \quad \mu^T\bar{R} \\
\text{subj. to} & \quad R \geq \bar{R}, R \in C(H, P) \\
\end{aligned} \tag{11}
\]

To solve the presented problems, we embed them in a higher dimensional space, leading to a formulation similar to that in [10], [12]. First we need the following definition.

**Definition 1:** Let \( \mathcal{G}(H) \) be the set that contains all feasible rate-power tuples for given channel realizations \( H \):

\[
\mathcal{G}(H) := \{(R, P) : R \in C(H, P)\}. \tag{12}
\]

Obviously the boundary of \( \mathcal{G}(H) \) are the points we are interested in, since they dominate all points in the interior. This means, for all \( (R, P) \) in the interior of \( \mathcal{G}(H) \) at least one component of \( R \) can be increased while all others remain fixed without leaving the set. In analogy to the single antenna case [14] we can state the following lemma.
Lemma 2: The set $\mathcal{G}(\mathbf{H})$ is a convex set and duality holds also for the expanded set.

The proof follows easily from the concavity of $f_{\mathbf{H}, \bullet}(\mathcal{I})$ over the set of positive semidefinite matrices $\mathbf{Q}$. Note that the lemma can be also immediately seen if the objective function and the affine trace constraint is made explicite by using Lemma 1. The boundary of the set $\mathcal{G}(\mathbf{H})$ for random channels $\mathbf{H}$ is depicted in Figure 1. The convexity is obvious. Note, that each horizontal slice represents the capacity region for a specific sum power $\bar{P}$.

![Fig. 1. Boundary of the region $\mathcal{G}(\mathbf{H})$ for an exemplary MIMO BC with $M = 2$ users, $U = V_1 = V_2 = 2$ antennas and random channel realizations.](image)

IV. MINIMUM SUM POWER

In this section we study Problem 2. There is a simple approach to solve this problem. Following the polymatroid structure of the MIMO OFDM MAC capacity region the problem can be written explicitely as

$$\begin{align*}
\min_{\mathcal{G}(\mathbf{H}), \mu \geq 0} & \quad P - \mu^T \mathbf{R} \\
\text{s.t.} & \quad f_{\mathbf{H}, \mathbf{Q}}(\mathcal{I}) \geq \bar{R}(\mathcal{I}), \quad \forall \mathcal{I} \subseteq \mathcal{M}.
\end{align*}$$

(13)

Note that the number of constraints is $2^M - 1$ since for each subset of users (in fact each face of the polymatroid) a constraint has to be met. This problem is a convex program and can be solved using standard convex optimization tools such as the YALMIP package [15]. Since the number of constraints grows exponentially with $M$, the problem becomes very complex even for moderate numbers of users.

In the following, a different approach is chosen. Equivalently to (13), the subsequent problem can be considered:

$$\begin{align*}
\min_{\mathcal{G}(\mathbf{H}), \mu \geq 0} & \quad P - \mu^T \mathbf{R} \\
\text{s.t.} & \quad \mathbf{R} \geq \bar{\mathbf{R}}
\end{align*}$$

(14)

Unfortunately, the notion of marginal utility functions introduced in [14] can not be applied to the MIMO (OFDM) BC. However, we will have a look at the objective function in (14). To this end we define

$$\tilde{R}_m(\mathbf{H}) = \left[ \arg \max_{\mathcal{G}(\mathbf{H})} \mu^T \mathbf{R} - P \right]_m$$

(15)

to be the $m$th rate-component of the optimizing set ($\bar{\mathbf{R}}, \bar{P}$) and propose Algorithm 1 to solve the minimum sum power problem.

**Algorithm 1 MIMO-OFDM Sum Power Minimization**

1. initialize $\mu^{(0)} = 0$
2. while desired accuracy not reached do
   1. find upper bound $\tilde{\mu}^{(i+1)}$ on $\mu^{(i+1)}$ such that $
      \tilde{R}_m(\mathbf{H}) > \tilde{R}_m$
   2. find $\mu^{(i+1)}$ by bisection such that
      $\tilde{R}_m(\mathbf{H}) = \tilde{R}_m$
3. end for
4. end while

Note that the desired rate vector $\bar{\mathbf{R}}$ might not be a vertex of the optimal polymatroid but may lie on the sum capacity plane (imagine the red circle in Figure 3 lying on the front plane). Then after convergence of the Algorithm 1 (which is indicated by very small variations of the weights), a linear system of $M!$ equations has to be solved determining the time-sharing factors $\{\alpha_i\}_{i=1}^{M!}$ with $\sum_i \alpha_i = 1$ corresponding to the $M!$ vertices (encoding orders) of the plane.

$$\bar{\mathbf{R}} = \alpha_1 \bar{R}_{\pi_1} + \ldots + \alpha_{M!} \bar{R}_{\pi_{M!}}$$

(16)

For this special case where only a sum rate constraint is active, further a fast iterative algorithm is presented in [9] based on an iterative water-filling procedure.

Interestingly, the optimal Lagrangian multipliers $\hat{\mu}$ reveal the necessary Dirty-Paper Precoding order for the MIMO OFDM BC. Since $\hat{\mu}$ constitutes the normal vector of a tangent hyperplane to the rate vector $\bar{\mathbf{R}}$, the optimal encoding order (and reverse decoding order in the dual MAC) $\pi$ is given by the ordering of the Lagrangian factors such that:

$$\hat{\mu}_{\pi(1)} \geq \ldots \geq \hat{\mu}_{\pi(M!)}.$$  

(17)

This is in analogy to the weighted rate sum maximization. The convergence properties of Algorithm 1 is discussed in the following section.

V. MINIMUM RATES

The developed methodology can be modified to guarantee minimum rates in the weighted sum rate maximization. In analogy to (10) and (14) Problem 3 can be embedded in the
enhanced set $\mathcal{G}(H)$. A reformulation is given by:

$$\min_{\mathcal{G}(H)} \lambda P - (\mu + \mu^*)^T R$$

subject to $R \geq \tilde{R}$

$$\lambda \geq 0, \mu \geq 0$$

Equation (18) reveals the intimate connection to Problem 2. However, the sum power is limited in this case and the initial weights $\mu$ constitute an additional value to the Lagrangian factors $\mu^*$. In analogy to (15) define

$$\tilde{R}_m(\mu^*, H) = \left[ \arg \max_{\nu(H), P} (\mu + \mu^*)^T R - \lambda P \right]_m$$

(19)

and, similarly, we can use Algorithm 2 for Problem 3.

Obviously, if the algorithm settles down and delivers some weight vector we have found a solution to the problem. Note, however, that (apart from the two user case and the single antenna case [11], [12]) convergence of the algorithm is not fully clear. In fact we need to establish the following:

Let $\mu^{(1)}, \mu^{(2)}$ be two weight vectors differing only in the $n$th component, i.e. $\mu_n^{(2)} > \mu_n^{(1)}$, $\mu_n^{(2)} = \mu_n^{(1)}$ for all $m \neq n$. Suppose that the optimizing rates are related by

$$\tilde{R}_m(\mu^{(2)}, H) \leq \tilde{R}_m(\mu^{(1)}, H) \quad \forall m \neq n$$

(20)

Then, assuming feasibility convergence follows from a simple monotonicity argument. At first, it is clear that by the convexity and by the fact that the user-wise extreme point are achieved on the coordinate axes that for $\mu_n^{(2)} > \mu_n^{(1)}$, and $\mu_n^{(2)} = \mu_n^{(2)}$, $m \neq n$, the rate $R_m$ must increase indicated by condition (21). More involved is condition (20). Here, observe that (while fixing the other components) the functions $R_n(R_m), n \neq n$, are monotonously decreasing and convex. Now, suppose for the moment differentiability. Since any point on the boundary of the region is the solution to $\max \mu^T R$ we have from the optimality conditions $\partial R_n/\partial R_m = -\mu_n/\mu_n$. Hence, if we increase $\mu_n$ while fixing the other weights we increase the slope in the regarded direction at the oscillation point of a tangent hyperplane with normal vector $\mu$. However, although quite intuitive, the trajectory on the boundary may not be such that the rates decrease. A rigorous proof has not yet been found.

The optimal Dirty-Paper-Encoding order is again a byproduct of the optimizing Lagrangian factors $\mu^*$: In contrast to Problem 2, now the ordering of the sum of weights $\mu$ and Lagrangian factors $\mu^*$ constitutes the optimal encoding order $\pi^*$:

$$\mu_{\pi(1)} + \mu^*_{\pi(1)} \geq \cdots \geq \mu_{\pi(M)} + \mu^*_{\pi(M)}.$$  

(24)

Note, that this fact was already observed for the OFDM case in [11].

Regarding computational complexity, step (3) of both algorithms although seeming simple contains a considerable computational challenge. In case of Algorithm 1 this problem can be solved by standard interior point methods while for Algorithm 2 the algorithm from [4] can be used. However, both Algorithm 1 and 2 consist of a sequence of convex optimization problems bringing along a high computational demand.

Recently, an interesting subgradient method was proposed in [8]. We can substantially modify this approach for the problem at hand [16]. Define the function

$$g(\mu^*) := \max_{R \in \mathcal{C}(H, P)} \mu^T R + \sum_{m=1}^M \mu^*_m (R_m - \tilde{R}_m)$$

for $\mu^*_m \geq 0$. It is clear that $g(\mu^*) \geq \max_{R \in \mathcal{C}(H, P)} \mu^T R$ for any such $\mu^*_m \geq 0$. Hence, we can minimize $g(\mu^*)$ over all possible $\mu^*$. To do so the ellipsoid method can be invoked. Let $\tilde{R}$ be the solution for $\tilde{R}$:

$$\tilde{R} = \arg \max_{R \in \mathcal{C}(H, P)} \mu^T R + \sum_{m=1}^M \mu^*_m (R_m - \tilde{R}_m) \leq \tilde{R}$$

Then a subgradient is easily found by observing that

$$g(\tilde{R}) \geq g(\mu^*) + \sum_{m=1}^M (\tilde{R}_m - \tilde{R}_m)(\tilde{R} - \tilde{R}_m)$$

Thus we can minimize the Lagrangian dual function. This approach will be intensively studied in [16].

Let us illustrate the presented algorithms. Figures 2 illustrates the convergence behavior of the Minimum Rates Algorithm for a random channel with $K = 1$ subcarrier, $U = 2$ antennas at the base station and $V_1 = V_2 = V_3 = 2$ antennas at each mobile. The required rates are set to $\tilde{R} = [2 0 1]^T$ bps/Hz. Figure 2 shows that after a moderate number of iterations the algorithm converges. Figure 3 illustrates the polyanomial with the rate tuple $\tilde{R} = [2 5.12 1]^T$ bps/Hz corresponding to the optimal covariance matrices. This vertex is achieved by the encoding order 2 → 1 → 3.

VI. CONCLUSIONS

We provided an analysis of resource allocation schemes for the MIMO OFDM Broadcast Channel. Complementary
to the weighted rate sum-maximization, we considered the problems of minimizing the sum power to achieve a certain set of rates and to maximize the weighted rate sum for a fixed power budget and minimum required rates. For both problems we presented algorithms. To this end we modified concepts known from the single antenna case. Further the optimal Dirty-Paper Encoding order was derived and shown to consist of the decreasing ordering of Lagrangian multipliers $\mu$ and in the first case and the decreasing ordering of the sum of Lagrangian multipliers and weights $(\mu + \mu^*)$ in the minimum rates case.

REFERENCES


