Second-Order Statistics of Equal-Gain and Maximal-Ratio Combining for the $\alpha - \mu$ (Generalized Gamma) Fading Distribution

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Abstract—This paper derives exact expressions for the level crossing rate and average fade duration of equal-gain and maximal-ratio combiners for the $\alpha - \mu$ (Generalized Gamma) fading Distribution, assuming $M$ unbalanced, non-identical, independent diversity channels. The analytical results are thoroughly validated by reducing the general case to some particular cases (Nakagami-$m$ and Weibull), for which the solutions are known, and by means of simulation, for the more general case. In addition, several new closed-form expressions are also obtained for other special cases.

Index Terms—Average fade duration, diversity-combining techniques, general fading distribution, level crossing rate

I. INTRODUCTION

In [1], [2], a physical fading model for the Generalized Gamma Distribution (GGD) was proposed. In its rewritten form [1], [2], this distribution was renamed the $\alpha - \mu$ Distribution so as to point up the physical parameters involved. The GGD was first proposed by Stacy [3] and in Stacy’s own words, the aim of his proposal “concerns a generalization of the Gamma Distribution”, which “in essence is accomplished by supplying a positive parameter included as an exponent to the Gamma Distribution”, which “in essence is accomplished by supplying a positive parameter included as an exponent in the exponential factor of the Gamma Distribution”. Stacy’s work was connected neither with any specific application nor with any physical modelling of any given phenomenon. It was purely a mathematical problem in which some statistical properties of a generalized version of the Gamma Distribution were investigated. The derivation of the $\alpha - \mu$ Distribution [1], [2], in contrast, has as its base a fading model. Thence its parameters are directly associated with the physical properties of the propagation medium. The Generalized Gamma or $\alpha - \mu$ Distribution is general, flexible, and has easy mathematical tractability. It includes important distributions such as Gamma (and its discrete versions Erlang and Central Chi-Squared), Nakagami-$m$ (and its discrete version Chi), Exponential, Weibull, One-Sided Gaussian, and Rayleigh. Its density, cumulative function, and moments appear in simple closed-form expressions. All these features combined make the Generalized Gamma Distribution very attractive. Using the fading model as proposed in [1], [2], a deeper characterization of the $\alpha - \mu$ Distribution can be achieved.

This paper derives exact expressions for the level crossing rate (LCR) and average fade duration (AFD) of equal-gain combining (EGC) and maximal-ratio combining (MRC) for channels undergoing the $\alpha - \mu$ fading statistics. The derivations assume unbalanced, non-identical, and independent branches. The general results are thoroughly validated by reducing them to particular cases (e.g., Nakagami-$m$ and Weibull), for which the analytical results are known, and by means of simulation, for the more general case. Some original exact and closed-form expressions are also obtained for other special cases of EGC and MRC.

II. THE $\alpha - \mu$ FADING MODEL

In this section, we revisit the $\alpha - \mu$ fading model as proposed in [1], [2]. In addition, based on this model, we obtain some higher order statistics necessary for the calculations that follow. The fading model for the $\alpha - \mu$ Distribution considers a signal composed of clusters of multipath waves propagating in a non-homogeneous environment. Within any one cluster, the phases of the scattered waves are random and have similar delay times with delay spreads of different clusters being relatively large. The clusters of multipath waves are assumed to have the scattered waves with identical powers. The resulting envelope is obtained as a non-linear function of the modulus of the sum of the multipath components. Such a non-linearity is manifested in terms of a power parameter, so that the resulting signal intensity is obtained not simply as the modulus of the sum of the multipath components, but as this modulus to a certain given exponent. Assuming that a certain given point the received signal encompasses an arbitrary number $n_i$ of multipath components, the resulting envelope $R_i$ at the $i$th branch, $i = 1, ..., M$, can be written as

$$R_i^{\alpha_i} = \sum_{j=1}^{n_i} (X_{ij}^2 + Y_{ij}^2)$$

where $X_{ij}$ and $Y_{ij}$ are zero-mean mutually independent Gaussian processes with identical variances $Var(X_{ij}) = Var(Y_{ij}) = \sigma_i^2 = \hat{r}_i^{\alpha_i}/2m_i$, $\alpha_i$ is the power parameter, $\hat{r}_i$ is the $\alpha_i$-root mean value of $R_i^{\alpha_i}$, i.e., $\hat{r}_i = \sqrt[\alpha_i]{E(R_i^{\alpha_i})}$. $E(\cdot)$ and $Var(\cdot)$ are the mean and variance operators, respectively. The corresponding probability density function (PDF) $f_{R_i}(r_i)$ of $R_i$, as given in [1], [2], is

$$f_{R_i}(r_i) = \frac{\alpha_i}{r_i^\alpha} \frac{\mu_i^{\alpha_i} \Gamma(\mu_i)}{\Gamma(\mu_i - 1)} e^{\left(-\mu_i \frac{r_i^{\alpha_i}}{\hat{r}_i^{\alpha_i}}\right)}$$

where $\mu_i > 0$ is the inverse of the normalized variance of $R_i^{\alpha_i}$, i.e.,

$$\mu_i = \frac{E^2(R_i^{\alpha_i})}{Var(R_i^{\alpha_i})}$$

and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function. More specifically, $\mu_i > 0$ is the real extension of $n_i$. For $\mu_i = 1,$
(2) reduces to the Weibull PDF, whereas for $\alpha_i = 2$ reduces to the Nakagami-m PDF. The cumulative distribution function (CDF) $F_{R_i}(r_i)$ of the envelope $R_i$ is given by

$$F_{R_i}(r_i) = \frac{\Gamma(\mu_i, \mu_i r_i^{\alpha_i}/\sigma_{R_i}^{\alpha_i})}{\Gamma(\mu_i)}$$  \hspace{1cm} (4)$$

where $\Gamma(z, y) = \int_0^y t^{z-1} e^{-t} dt$ is the incomplete Gamma function. For isotropic scattering, the time derivative $\dot{X}_i$ and $\dot{Y}_i$ of $X_i$ and $Y_i$, respectively, are known to be zero-mean Gaussian variates with variances $\sigma_i^2 = 2\pi f_m^2 \sigma_i^2$ [4], where $f_m$ is the maximum Doppler shift. Correspondingly, from (1), the conditional PDF of the time derivative of $R_i$, denoted as $\dot{R}_i$, given $R_i$, is easily found to be

$$f_{\dot{R}_i|R_i}(\dot{r}_i|r_i) = \frac{1}{\sqrt{2\pi} \sigma_{\dot{R}_i}} \exp \left( -\frac{1}{2} \frac{\dot{r}_i^2}{\sigma_{\dot{R}_i}^2} \right)$$  \hspace{1cm} (5)$$
in which $\sigma_{\dot{R}_i}^2 = \frac{\sigma_i^2}{\alpha_i^2 \mu_i}, \Omega_i 4\pi f_m^2$ (see Appendix), where $\Omega_i = \frac{\alpha_i}{\alpha_i^2 \mu_i}$.

III. LCR and AFD

The LCR is defined as the average number of times a fading signal crosses a given signal level in the negative (or positive) direction within a certain period of time. Denoting the time derivative of the envelope $R$ as $\dot{R}$ and the crossing level as $r$, the LCR is estimated as

$$N_R(r) = \int_0^\infty \dot{r} f_{\dot{R},R}(r, \dot{r}) d\dot{r}$$  \hspace{1cm} (6)$$

where $f_{\dot{R},R}(\cdot, \cdot)$ is the joint PDF of $R$ and $\dot{R}$. The AFD is defined as the mean time the received envelope is found below the given threshold $r$ and is formulated as

$$T_R(r) = \frac{F_R(r)}{N_R(r)}$$  \hspace{1cm} (7)$$

where $F_R(\cdot)$ is the CDF of $R$. In the following, it shall be taken as the combiner output, and (6) and (7) shall be used to derive the LCR and AFD of EGC and MRC for $M$ unbalanced, non-identical, and independent $\alpha - \mu$ fading channels.

A. Equal-Gain Combining

In EGC, the received signals with envelopes $R_i$ are cophased and added so that the combiner output envelope $R$, already taking into account the resultant noise power at the combiner output, is written as

$$R = \frac{1}{\sqrt{M}} \sum_{i=1}^M R_i$$  \hspace{1cm} (8)$$

Deriving both sides of the equality above, the time derivative $\dot{R}$ of $R$ is given by

$$\dot{R} = \frac{1}{\sqrt{M}} \sum_{i=1}^M \dot{R}_i$$  \hspace{1cm} (9)$$

The CDF of $R$ can be calculated by integrating the joint PDF of $R_i$ over the $M$-dimensional volume bounded by the hyperplane $\sqrt{M} r = \sum_{i=1}^M r_i$. Then, using a similar procedure applied in [5], we obtain

$$F_R(r) = \int_0^{\sqrt{M} r} \cdots \int_0^{\sqrt{M} r - r_{M-1}} \int_0^{\sqrt{M} r - \sum_{i=1}^{M-2} r_i} \cdots \int_0^{\sqrt{M} r - \sum_{i=1}^{M-1} r_i} \cdots \int_0^{\sqrt{M} r - \sum_{i=1}^M r_i}$$

$$\times \prod_{i=1}^M f_{R_i}(r_i) dr_1...dr_{M-1} dr_M$$  \hspace{1cm} (10)$$

where $f_{R_i}(r_i)$ is given by (2). From (5), (8), and (9), we note that $f_{R|R_1,\ldots,R_M}(\cdot, \ldots, \cdot)$ is zero-mean Gaussian distributed with variance $\sigma^2_R = \sum_{i=1}^M \sigma_i^2 / M$. Deriving (10) with respect to $r$ in order to obtain $f_{\dot{R}}(r)$ and using the Bayes’ rule, $f_{R_1,\ldots,R_M,\dot{R}}(\cdot, \ldots, \cdot)$ can be found as (12), where $f_{R_1,\ldots,R_M,\dot{R}}(\cdot, \ldots, \cdot)$ is the joint PDF of $R_i$ and $\dot{R}$. Then, by means of the properties of the conditional probability, the following can be written

$$f_{R_1,\ldots,R_M,\dot{R}}(r_1, \ldots, r_M, \dot{r}) = f_{R_1,\ldots,R_M}(r_1, \ldots, r_M) \times f_{\dot{R}|R_1,\ldots,R_M}(\dot{r}|r_1, \ldots, r_M)$$  \hspace{1cm} (11)$$

B. Maximal-Ratio Combining

In MRC, the received signals are cophased, each signal is amplified appropriately for optimal combining, and the resultant signals are added so that the combiner output envelope $\dot{R}$ is given by

$$r = \sum_{i=1}^M R_i^2$$  \hspace{1cm} (12)$$

The time derivative $\dot{R}$ of $R$ can be expressed as

$$\dot{R} = \sum_{i=1}^M \frac{R_i}{\sqrt{M}} \dot{R}_i$$  \hspace{1cm} (13)$$

The MRC analysis follows exactly the same steps as previously detailed for EGC. However, the hyperplane used to compute $F_R(\cdot)$ is $r^2 = \sum_{i=1}^M r_i^2$. Therefore, the resulting variance of $\dot{R}$ can be expressed as $\sigma^2_{\dot{R}} = \sum_{i=1}^M R_i^2 \sigma_i^2 / R$. The resulting $F_R(\cdot)$, $f_{\dot{R},R}(\cdot, \cdot)$, and $N_R(\cdot)$ are given by (16), (17), and (18), respectively.

$$F_R(r) = \int_0^{r} \cdots \int_0^{\sqrt{M} r - \sum_{i=1}^{M-2} r_i} \cdots \int_0^{\sqrt{M} r - \sum_{i=1}^{M-1} r_i} \cdots \int_0^{\sqrt{M} r - \sum_{i=1}^M r_i}$$

$$\times \prod_{i=1}^M f_{R_i}(r_i) dr_1...dr_{M-1} dr_M$$  \hspace{1cm} (14)$$

From (18), (16), and (7), the AFD follows directly. To our best knowledge, these expressions are new.
\[ f_{R,R}(r, \tilde{r}) = \sqrt{M} \int_0^{\sqrt{M}r} \int_0^{\sqrt{M}r-Mr_1} \ldots \int_0^{\sqrt{M}r-Mr_{M-1}} \sqrt{r^2 - \sum_{i=2}^{M-1} r_i} \]
\[ \times f_{R_1,R_2,\ldots,R_M,\tilde{R}} \left( \sqrt{M}r - \sum_{i=2}^{M} r_i, r_2, \ldots, r_M, \tilde{r} \right) \ dr_2 \ldots dr_{M-1} dr_M \]  
(12)

\[ N_R(r) = \sqrt{2\pi} f_m \int_0^{\sqrt{M}r} \int_0^{\sqrt{M}r-Mr_1} \ldots \int_0^{\sqrt{M}r-Mr_{M-1}} \sqrt{r^2 - \sum_{i=2}^{M-1} r_i} \]
\[ \times f_{R_1} \left( \sqrt{M}r - \sum_{i=2}^{M} r_i \right) \prod_{i=2}^{M} f_{R_i}(r_i) \ dr_2 \ldots dr_{M-1} dr_M \]  
(13)

\[ f_{R,R}(r, \tilde{r}) = \int_0^{r} \int_0^{\sqrt{r^2-Mr^2}} \ldots \int_0^{\sqrt{r^2-\sum_{i=2}^{M-1} r_i}} \sqrt{r^2 - \sum_{i=2}^{M} r_i} \]
\[ \times f_{R_1,R_2,\ldots,R_M,\tilde{R}} \left( r^2 - \sum_{i=2}^{M} r_i^2, r_2, \ldots, r_M, \tilde{r} \right) \ dr_2 \ldots dr_{M-1} dr_M \]  
(17)

\[ N_R(r) = \sqrt{2\pi} f_m \int_0^{r} \int_0^{\sqrt{r^2-Mr^2}} \ldots \int_0^{\sqrt{r^2-\sum_{i=2}^{M-1} r_i}} 1 \]
\[ \times f_{R_1} \left( r^2 - \sum_{i=2}^{M} r_i^2 \right) \prod_{i=2}^{M} f_{R_i}(r_i) \ dr_2 \ldots dr_{M-1} dr_M \]  
(18)

C. Special Cases

As commented before, the \( \alpha - \mu \) fading model is a general fading model that includes as special cases Nakagami-\( m \) and Weibull. By setting \( \mu_i = 1 \) (Weibull case), our results reduce to those of [6]. In the same way, by setting \( \alpha_i = 2 \) (Nakagami-\( m \) case) and \( \Omega_i = \Omega \), again our formulations reduce in an exact manner to those obtained in [5, Eqs. 23-24] for EGC scheme and in [5, Eqs. 36-37] for MRC one. In particular, for \( \alpha_i = 2 \) and unbalanced case, the expressions derived here reduce to those presented in [7, Eqs. 29-30] (MRC case) and in [7, Eqs. 39-40] (EGC case) for Nakagami-\( m \) fading channels. In the same way, for MRC, \( M = 2, \mu = 1 \), and \( \alpha = 4 \) our formulations reduce to those of [6, Eqs. 16-17].

Some new closed-form expressions are obtained for balanced branches and same fading parameters, i.e., \( \alpha_i = \alpha \) and \( \mu_i = \mu \). Therefore, it follows that

- For EGC and \( \alpha = 1 \), from (7), (10), and (13), and after simple algebraic manipulations

\[ N_R(r) = \sqrt{2\pi} f_m \frac{M^{2\mu-1}}{\Gamma(M\mu)} \left( \frac{\mu r}{\Omega} \right)^{M\mu-\frac{1}{2}} \exp \left( -\sqrt{M\mu} \frac{r}{\Omega} \right) \]  
(19)

\[ T_R(r) = \frac{1}{\sqrt{2\pi} f_m} \frac{\mu r}{\mu + \frac{1}{2}} \frac{1}{\Gamma(\mu + \frac{1}{2})} \exp \left( -\sqrt{\frac{\mu r}{\Omega}} \right) \]  
(20)

- For EGC, \( M = 2 \), and \( \alpha = 2 \), from (7), (10), and (13), and following the same steps as in [7, App. B]

\[ N_R(r) = 2 \sqrt{2\pi} f_m \frac{\mu^2}{\mu + \frac{1}{2}} \frac{1}{\Gamma(\mu + \frac{1}{2})} \exp \left( -2 \mu r^2 / \Omega \right) \]  
(21)

where \( _1F_1(\cdot; \cdot; \cdot) \) denotes the confluent hypergeometric function [8, Eq. 9.210.1].
• For MRC, $M = 2$, and $\alpha = 4$, from (7), (16), and (18), and following the same steps as in [7, App. B]

$$N_R(r) = \frac{\sqrt{\pi} f_{in} \mu^{2\mu-\frac{1}{2}} \Gamma \left(\mu + \frac{1}{2}\right) s^{\mu-2}}{2^{2\mu-2} \Omega^{2\mu-\frac{1}{2}} \Gamma(\mu)} \exp \left(-\frac{\mu r^4}{\Omega}\right)$$

$$\times F_1 \left(2\mu; 2\mu + 1; \frac{1}{2} \frac{\mu r^4}{2\Omega}\right)$$ (22)

IV. NUMERICAL RESULTS

In this section, some plots illustrate the expressions obtained here. In addition, the validity of these expressions is checked by comparing the theoretical curves against the simulation results. As will be observed, an excellent agreement has been achieved between the simulation and the analytical results. Figs. 1, 2, 3, and 4 depict the normalized LCR (left axis), $N_R(r)/f_{in}$, and AFD (right axis), $T_R(r)f_{in}$, as a function of the normalized envelope $\rho = r/\sqrt{\Omega}$, for EGC and MRC. The branches are assumed to be balanced ($\Omega = \Omega$) and having arbitrary $\alpha - \mu$ parameters. The plots are exemplified for $M = 2$ (Figs. 1 and 2) and $M = 4$ (Figs. 3 and 4) diversity branches. The curves with no diversity have been omitted for the sake of clarity. Of course, as well-known, the use of diversity reduces drastically the deleterious effect of the fading. Note that EGC and MRC yield fairly similar performances. By comparing Figs. 1 and 2 against Figs. 3 and 4, it can be seen that, for the same fading condition, as a consequence of the improvement of the output signal with the increase of the number of branches, lower levels are crossed at lower rates whereas higher levels are crossed at higher rates.

For $\mu_1 = \mu = 2$ and several values of $\alpha_1 = \alpha$, Figs. 5 and 6 depict the simulation curves of the normalized LCR, as a function of the normalized envelope, for EGC and MRC, respectively, using two balanced branches. Note the excellent agreement between the simulated and theoretical curves. For other fading conditions, exhaustive simulations have been performed and, in all of the cases, a very good adjustment has been achieved.

V. CONCLUSIONS

In this paper, exact formulas for LCR and AFD using EGC and MRC techniques with M unbalanced, non-identical, and independent branches undergoing the $\alpha - \mu$ fading statistics have been obtained. The formulas have been validated by reducing the general case to some special cases for which the solutions are known and, more generally, by means of simulation. Additionally, some original exact and closed-form expressions were obtained for other special cases of EGC and MRC.

REFERENCES

The aim of this Appendix is to find the variance of the conditional PDF of $\hat{R}_i$ given $R_i$. From (1), we have

$$R_i^{\alpha_i} = \sum_{l=1}^{n_i} (X_{il}^2 + Y_{il}^2)$$

(23)

Derivating each side of the equality in (23)

$$\alpha_i R_i^{\alpha_i - 1} \frac{d}{d\hat{R}_i} \hat{R}_i = \sum_{l=1}^{n_i} 2X_{il} \hat{X}_{il} + 2Y_{il} \hat{Y}_{il}$$

(24)

Rearranging the terms and applying the conditional variance operator

$$\sigma_{R_i}^2 = \sum_{l=1}^{n_i} \frac{4X_{il}^2}{\alpha_i^2 R_i^{2\alpha_i - 2}} \sigma_i^2 + \frac{4Y_{il}^2}{\alpha_i^2 R_i^{2\alpha_i - 2}} \sigma_i^2$$

(25)

Knowing that $\sigma_i^2 = 2\pi^2 f_m^2 \sigma_i^2 = \pi^2 f_m^2 \Omega_i/\mu_i$, we obtain

$$\sigma_{R_i}^2 = \sum_{l=1}^{n_i} \frac{(X_{il}^2 + Y_{il}^2)}{\alpha_i^2 R_i^{2\alpha_i - 2}} \Omega_i$$

(26)

Substituting (23) into (26) we conclude the demonstration.